HALL INVARIANTS, HOMOLOGY OF SUBGROUPS, AND CHARACTERISTIC VARIETIES

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Abstract. Given a finitely-generated group G, and a finite group Γ, Philip Hall defined $\delta_\Gamma(G)$ to be the number of factor groups of G that are isomorphic to Γ. We show how to compute the Hall invariants by cohomological and combinatorial methods, when G is finitely-presented, and Γ belongs to a certain class of metabelian groups. Key to this approach is the stratification of the character variety, $\operatorname{Hom}(G, \mathbb{K}^*)$, by the jumping loci of the cohomology of G, with coefficients in rank 1 local systems over a suitably chosen field $\mathbb{K}$. Counting relevant torsion points on these "characteristic" subvarieties gives $\delta_\Gamma(G)$. In the process, we compute the distribution of prime-index, normal subgroups $K \triangleleft G$ according to $\dim_{\mathbb{K}} H_1(K;\mathbb{K})$, provided $\operatorname{char}\mathbb{K} \neq |G : K|$. In turn, we use this distribution to count low-index subgroups of G. We illustrate these techniques in the case when G is the fundamental group of the complement of an arrangement of either affine lines in $\mathbb{C}^2$, or transverse planes in $\mathbb{R}^4$.

Contents

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1. Introduction

1.1. **Hall invariants.** In [18], Philip Hall introduced several notions in group theory. Given a finitely-generated group G, and a finite group Γ, he defined $\delta_\Gamma(G)$ to be the number of surjective representations of G to Γ, up to automorphisms of Γ:

$$\delta_\Gamma(G) = |\mathrm{Epi}(G,\Gamma)/\operatorname{Aut}\Gamma| . \tag{1.1}$$

In other words, the *Hall invariant* $\delta_\Gamma(G)$ counts all factor groups of G that are isomorphic to Γ. When $G = \pi_1(X)$ is the fundamental group of a connected 2-complex X with finite 1-skeleton, $\delta_\Gamma(\pi_1(X))$ counts all connected, regular covers of X with deck transformation group Γ.

Suppose G has a finite presentation, with generators $x_1, \dots, x_n$, and relators $r_1, \dots, r_m$. The Hall invariant $\delta_\Gamma(G)$ can be computed by counting generating sets of Γ that have size n and satisfy the relations r_j in Γ, and then dividing the result by the order of $\mathrm{Aut}(\Gamma)$. While this method can be implemented on a computer algebra system like *GAP* [15][1], the computation breaks down even for moderately large n and $|\Gamma|$. One of the purposes of this paper is to show how the Hall invariants $\delta_\Gamma(G)$ can be computed more efficiently, by combinatorial and homological methods, at least when Γ belongs to a certain class of finite metabelian groups.

1.2. **Abelian representations.** In Section 2, we start by recalling a formula of Hall [18]. Using what he called the *Eulerian function* of the finite group Γ, together with Möbius inversion, Hall showed that $\delta_\Gamma(G) = \frac{1}{|\operatorname{Aut}\Gamma|} \sum_{H\le\Gamma} \mu(H)\, |\mathrm{Hom}(G,H)|$, where μ is the Möbius function of the subgroup lattice of Γ. For a p-group Γ, the Möbius and Eulerian functions were computed by Weisner [43].

In Section 3, we use these results of Hall and Weisner, together with a result from Macdonald's book [30], to arrive at a completely explicit formula for $\delta_\Gamma(G)$ in the simplest case: that of a finite abelian group Γ. The expression for $\delta_\Gamma(G)$, given in Theorem 3.1, depends only on Γ and the abelian factors of G. A similar expression was obtained (by other means) in [22], in the particular case $G = F_n$.

1.3. **Metabelian representations.** The next level of difficulty in computing Γ-Hall invariants is presented by (split) metabelian groups Γ. Suppose $\Gamma = B \rtimes_\sigma C$ is a semidirect product of abelian groups, with monodromy homomorphism $\sigma : C \to \mathrm{Aut}(B)$. An epimorphism $\lambda : G \twoheadrightarrow \Gamma$ may be thought of as the lift of an epimorphism $\rho : G \twoheadrightarrow C$. As explained in [28], the lifts of a fixed homomorphism $\rho : G \to C$ are parametrized by $H^1(G, B_\rho)$, the first cohomology group of G with coefficients in the G-module $B = B_\rho$, with action $g \cdot b = \sigma(\rho(g))(b)$.

[1]The *GAP* command that computes $\delta_\Gamma(G)$ is `GQuotients(G,Γ)`. We were led to consider the Hall invariants after reading about this command in the *GAP* manual.

We consider split metabelian groups $\Gamma = B \rtimes_\sigma C$ for which B is an elementary abelian group, and C is a finite cyclic group. If $\mathbb{K}$ is a finite field with additive group B, then $H^1(G, B_\rho)$ may be identified with $H^1(G, \mathbb{K}_\rho)$ where again G acts on $\mathbb{K}$ by means of ρ. Now Shapiro's Lemma identifies the twisted cohomology group $H^1(G, \mathbb{K}_\rho)$ with the untwisted cohomology group $H^1(K_\rho, \mathbb{K})$, where $K_\rho = \ker \rho$. We are thus led to investigate the homology of finite-index, normal subgroups of G.

1.4. **Homology of finite-index subgroups.** Let G be a finitely-presented group, and $K \triangleleft G$ a normal subgroup. Assume the quotient group, $\Gamma = G/K$, is finite. A procedure to compute $H_1(K, \mathbb{Z})$ from a presentation of G and the coset representation of G on G/K was given by Fox [13]. The efficiency of Fox's method decreases rapidly with the increase in the index $|\Gamma| = |G : K|$. In Section 4, we overcome this problem, at least partially. Our approach (similar to that of Hempel [19, 20] and Sakuma [39]) is based on the representation theory of Γ, over suitably chosen fields $\mathbb{K}$.

Consider the homology group $H_1(K, \mathbb{K})$, and set $b_1^{(q)}(K) := \dim_{\mathbb{K}} H_1(K, \mathbb{K})$, where $q = \operatorname{char} \mathbb{K}$. The idea is to break $H_1(K, \mathbb{K})$ into a direct sum, according to the decomposition of the group algebra $\mathbb{K}\Gamma$ into irreducible representations. In order for this to work, we need the field $\mathbb{K}$ to be "sufficiently large" with respect to Γ; that is, q should not to divide $|\Gamma|$, and $\mathbb{K}$ should contain all roots of unity of order equal to the exponent of Γ. Let $\lambda : G \twoheadrightarrow \Gamma$ be an epimorphism, with kernel $K = K_\lambda$. In Theorem 4.6, we prove:

$$b_1^{(q)}(K_\lambda) = b_1^{(q)}(G) + \sum_{\rho \neq \mathbf{1}} n_\rho(\operatorname{corank} J^{\rho\circ\lambda} - n_\rho), \tag{1.2}$$

where ρ runs through all non-trivial, irreducible representations of Γ over the field $\mathbb{K}$, and $J^{\rho\circ\lambda}$ is the Jacobian matrix of Fox derivatives of the relators, $J = J_G$, followed by the representation $\rho \circ \lambda : G \to \mathrm{GL}(n_\rho, \mathbb{K})$.

When Γ is abelian and $\mathbb{K} = \mathbb{C}$, we recover from (1.2) a well-known result of Libgober [23] and Sakuma [39]: $b_1(K_\lambda) = b_1(G) + \sum_{\rho \neq \mathbf{1}}(\operatorname{corank} J^{\rho\circ\lambda} - 1)$, where ρ runs through all non-trivial, irreducible, complex representations of Γ. For other choices of $\mathbb{K}$, formula (1.2) gives information about the q-torsion coefficients of $H_1(K, \mathbb{Z})$, provided $q \nmid |\Gamma|$.

1.5. **Torsion points on characteristic varieties.** The next step is to interpret formula (1.2) in terms of the "Alexander stratification" of the character variety of G. This can be done for an arbitrary finite abelian group Γ, but, for simplicity, we restrict our attention to the case when Γ is cyclic, which is enough for our purposes here.

In Section 5, we start by reviewing the pertinent material on Alexander ideals and their associated varieties, in a more general context than usual. The Alexander

matrix, A_G, is the abelianization of the Fox Jacobian, J_G. The *d-th characteristic variety*, $V_d(G,\mathbb{K})$, is the subvariety of $\operatorname{Hom}(G,\mathbb{K}^*)$ defined by the codimension d minors of A_G. It can be shown that $V_d(G,\mathbb{K})\setminus\{\mathbf{1}\}$ is the set of non-trivial characters $\mathbf{t}\in\operatorname{Hom}(G,\mathbb{K}^*)$ for which $\dim_{\mathbb{K}} H^1(G,\mathbb{K}_{\mathbf{t}})\geq d$, see [21] and Remark 5.4.

In Section 6, we study the relationship between torsion points on the characteristic varieties of G and the homology of finite-index, normal subgroups $K\triangleleft G$. As mentioned above, we only consider the case when $\Gamma=G/K$ is cyclic, say $\Gamma=\mathbb{Z}_N$. In Theorem 6.2, we prove:

$$b_1^{(q)}(K_\lambda)=b_1^{(q)}(G)+\sum_{1\neq k\mid N}\phi(k)d_{\mathbb{K}}(\lambda^{N/k}). \tag{1.3}$$

where $d_{\mathbb{K}}(\mathbf{t})=\max\{d\mid \mathbf{t}\in V_d(G,\mathbb{K})\}$ is the *depth* of the character $\mathbf{t}\in\operatorname{Hom}(G,\mathbb{K}^*)$ with respect to the Alexander stratification. In particular, if $N=p$ is prime, then:

$$b_1^{(q)}(K_\lambda)=b_1^{(q)}(G)+(p-1)d_{\mathbb{K}}(\lambda). \tag{1.4}$$

In view of (1.4), we define $\beta^{(q)}_{p,d}(G)$ to be the number of index p, normal subgroups $K\triangleleft G$ for which $b_1^{(q)}(K)=b_1^{(q)}(G)+(p-1)d$. In Theorem 6.5, we prove:

$$\beta^{(q)}_{p,d}(G)=\tfrac{1}{p-1}\left|\operatorname{Tors}_{p,d}(G,\mathbb{K})\setminus\operatorname{Tors}_{p,d+1}(G,\mathbb{K})\right|, \tag{1.5}$$

where $\mathbb{K}=\mathbb{C}$ if $q=0$, and $\mathbb{K}=\mathbb{F}_{q^s}$ ($s=$ order of q in $\mathbb{F}_p^*$) if $q\neq 0$, and $\operatorname{Tors}_{p,d}(G,\mathbb{K})$ is the set of characters in $V_d(G,\mathbb{K})$ of order exactly p.

1.6. **Metabelian Hall invariants and low-index subgroups.** Once this is done, we are ready to return to the Hall invariants of G. In Section 7, we compute $\delta_\Gamma(G)$, for split metabelian groups Γ of the form $M_{p,q^s}=\mathbb{Z}_q^s\rtimes_\sigma\mathbb{Z}_p$, where p and q are distinct primes, $s=\operatorname{ord}_p(q)$, and σ has order exactly p. Examples are the dihedral groups $D_{2p}=M_{2,p}$ and the alternating group $A_4=M_{3,4}$. In Theorem 7.7, we prove:

$$\delta_{M_{p,q^s}}(G)=\tfrac{p-1}{s(q^s-1)}\sum_{d\geq 1}\beta^{(q)}_{p,d}(G)(q^{sd}-1). \tag{1.6}$$

This generalizes a result of Fox [14], who was the first to use Alexander matrices for counting metacyclic representations of fundamental groups of knots and links. Put together, formulas (1.6) and (1.5) express the Hall invariant $\delta_{M_{p,q^s}}(G)$ in terms of the number of p-torsion points on the Alexander strata of the character variety $\operatorname{Hom}(G,\mathbb{F}_{q^s})$.

In Section 8, we use formula (1.6), together with several formulas from §§2–3, to derive information about the number, $a_k(G)$, of index k subgroups of G. It was Marshall Hall [17] who showed how to compute these numbers recursively, in terms

of $|\mathrm{Hom}(G, S_l)|$, $1 \leq l \leq k$. Applying this method, we obtain (in Theorem 8.2):

$$a_3(G) = \tfrac{1}{2}(3^n - 1) + \tfrac{3}{2} \sum_{d \geq 1} \beta^{(3)}_{2,d}(G)(3^d - 1), \tag{1.7}$$

where $n = b^{(3)}_1(G)$. We also give formulas of this sort for the number, $a^{\triangleleft}_k(G)$, of index k, normal subgroups of G, provided $k \leq 15$ and $k \neq 8$ or 12.

1.7. **Arrangement groups.** We conclude with some explicit examples and computations in the case when G is the fundamental group of the complement of a subspace arrangement. This is meant to illustrate the theory developed so far, in a setting where topology and combinatorics are closely intertwined.

In Section 9, we look at complex hyperplane arrangements. By the Lefschetz-type theorem of Hamm and Lê, it is enough to consider arrangements of affine lines in $\mathbb{C}^2$. If G is the group of such an arrangement, the characteristic varieties $V_d(G, \mathbb{C})$ are well understood: they consist of subtori of the character torus, possibly translated by roots of unity. Furthermore, the tangent cone at the origin to $V_d(G, \mathbb{C})$ coincides with the "resonance" variety $R_d(G, \mathbb{C})$, which is determined by the combinatorics of the arrangement. The components of $V_d(G, \mathbb{C})$ not passing through the origin, though, are not *a priori* combinatorially determined. Their appearance affects the torsion coefficients in the homology of certain finite abelian covers of the complement, as we show in Example 9.3.

In Section 10, we turn to real arrangements. More precisely, we consider arrangements of transverse planes through the origin of $\mathbb{R}^4$. If G is the group of such a 2-*arrangement*, the varieties $V_d(G, \mathbb{C})$ need not be unions of translated subtori, as shown in [32], and also here, in Example 10.3. Furthermore, the tangent cone at the origin to $V_d(G, \mathbb{C})$ may not coincide with the resonance variety $R_d(G, \mathbb{C})$, as we point out in Remark 10.4. Finally, using the metabelian Hall invariants δ_{S_3} and δ_{A_4}, we recover the homotopy-type classification of complements of 2-arrangements of $n \leq 6$ planes in $\mathbb{R}^4$ (first established in [32]), and extend it to horizontal arrangements of $n = 7$ planes.

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2. Eulerian functions and Hall invariants

We start by reviewing two basic notions introduced by Philip Hall in [18]: the Eulerian function, $\phi(\Gamma, n)$, of a finite group Γ, and the Hall invariants, $\delta_\Gamma(G)$, of a finitely-generated group G.

2.1. **Eulerian function.** Let Γ be a finite group. The *Eulerian function* of Γ is defined as

$$\phi(\Gamma, n) = \#\{\text{ordered } n\text{-tuples } (g_1, \dots, g_n) \text{ that generate } \Gamma\}, \tag{2.1}$$

where repetitions among the g_i's are allowed. For example, $\phi(\mathbb{Z}_k, 1) = \phi(k)$, the usual Euler totient function.

Let $L(\Gamma)$ be the lattice of subgroups of Γ, ordered by inclusion. Let $\mu : L(\Gamma) \to \mathbb{Z}$ be the *Möbius function*, defined inductively by $\mu(\Gamma) = 1$, $\sum_{H \le K} \mu(H) = 0$. Then, the Eulerian function of Γ is given by:

$$\phi(\Gamma, n) = \sum_{H \le \Gamma} \mu(H) \, |H|^n \,, \tag{2.2}$$

see [18], and also [5] for a recent account.

The Eulerian function and the Möbius function of a finite p-group were computed by Weisner in [43]. To formulate Weisner's results, recall that the *Frattini subgroup* of a finite group Γ, denoted $\operatorname{Frat}\Gamma$, is the intersection of all maximal, proper subgroups of Γ. If Γ is a p-group, then $\operatorname{Frat}\Gamma = [\Gamma, \Gamma] \cdot \Gamma^p$, by the Burnside Basis Theorem (cf. [37, p. 140]).

According to Weisner, the Möbius function of a finite p-group Γ is given by:

$$\mu(H) = \begin{cases} (-1)^d p^{\frac{d(d-1)}{2}}, \text{ where } p^d = |\Gamma : H| & \text{if } \operatorname{Frat}\Gamma \le H, \\ 0 & \text{if } \operatorname{Frat}\Gamma \not\le H. \end{cases} \tag{2.3}$$

Now set $p^r = |\Gamma|$ and $p^s = |\Gamma : \operatorname{Frat}\Gamma|$. The Eulerian function of Γ is then given by:

$$\phi(\Gamma, n) = p^{(r-s)n} \prod_{i=0}^{s-1} (p^n - p^i). \tag{2.4}$$

2.2. **Hall invariants.** Let G be a finitely-generated group, and Γ a finite group. Let $\sigma_\Gamma(G) = |\operatorname{Hom}(G, \Gamma)|$ be the number of homomorphisms $G \to \Gamma$, and $\phi_\Gamma(G) = |\operatorname{Epi}(G, \Gamma)|$ the number of epimorphisms $G \twoheadrightarrow \Gamma$. The relation between σ and ϕ is given by Hall's enumeration principle:

$$\sigma_\Gamma(G) = \sum_{H \le \Gamma} \phi_H(G), \tag{2.5}$$

or, by Möbius inversion:

$$\phi_\Gamma(G) = \sum_{H \le \Gamma} \mu(H) \sigma_H(G). \tag{2.6}$$

Definition 2.3. Let G be a finitely-generated group. Let Γ be a finite group, with automorphism group $\operatorname{Aut}\Gamma$. The Γ-*Hall invariant* of G is $\delta_\Gamma(G) = \phi_\Gamma(G) / |\operatorname{Aut}\Gamma|$.

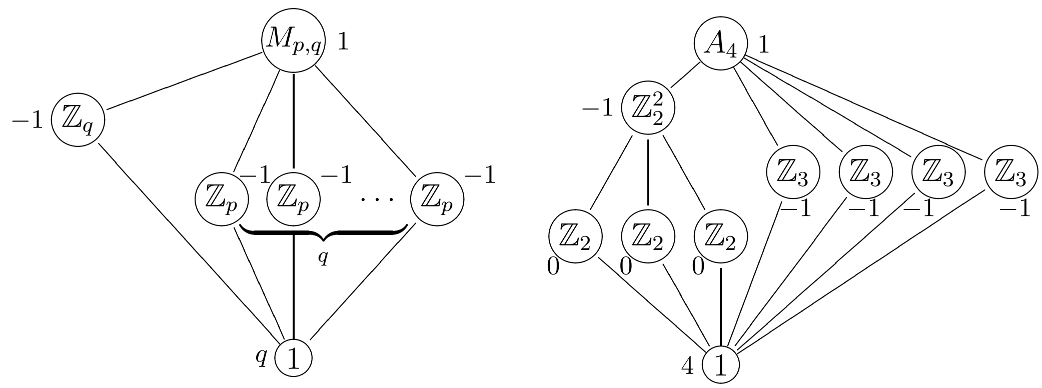

FIGURE 1. The subgroup lattice and Möbius function of $M_{p,q}$ and A_4

Since $\operatorname{Aut}\Gamma$ acts freely and transitively on $\operatorname{Epi}(G,\Gamma)$, the number $\delta_\Gamma(G)$ is an integer, which counts epimorphisms $G \twoheadrightarrow \Gamma$, up to automorphisms of Γ. In other words, $\delta_\Gamma(G)$ is the number of homomorphs of G that are isomorphic to Γ.

Note that

$$\phi_{\Gamma_1\times\Gamma_2}(G) = \phi_{\Gamma_1}(G)\phi_{\Gamma_2}(G) \tag{2.7}$$

whenever Γ_1 and Γ_2 are finite groups, with $(|\Gamma_1|, |\Gamma_2|) = 1$. In that situation, we also have $\operatorname{Aut}(\Gamma_1 \times \Gamma_2) = \operatorname{Aut}(\Gamma_1) \times \operatorname{Aut}(\Gamma_2)$, and so $\delta_{\Gamma_1\times\Gamma_2}(G) = \delta_{\Gamma_1}(G)\delta_{\Gamma_2}(G)$.

Now let $G = F_n$, the free group of rank n. Clearly, $\sigma_\Gamma(F_n) = |\Gamma|^n$ and $\phi_\Gamma(F_n) = \phi(\Gamma, n)$. Hence, by (2.2):

$$\delta_\Gamma(F_n) = \frac{\sum_{H\le\Gamma} \mu(H)\,|H|^n}{|\operatorname{Aut}(\Gamma)|}. \tag{2.8}$$

Example 2.4. Let $M_{p,q} = \mathbb{Z}_q \rtimes \mathbb{Z}_p$ be the metacyclic group of order pq, where p and q are primes, with $p \mid (q-1)$. Its subgroup lattice and Möbius function are shown in Figure 1. The automorphism group of $M_{p,q}$ is isomorphic to $M_{q-1,q} \cong \mathbb{Z}_q \rtimes \operatorname{Aut}(\mathbb{Z}_q)$, the holomorph of $\mathbb{Z}_q$. By (2.8):

$$\delta_{M_{p,q}}(F_n) = \frac{(p^n-1)(q^{n-1}-1)}{q-1}. \tag{2.9}$$

Example 2.5. Let A_4 be the alternating group on 4 symbols. The Möbius function is given in Figure 1. Furthermore, $\operatorname{Aut}(A_4) \cong S_4$, the symmetric group on 4 symbols. We get:

$$\delta_{A_4}(F_n) = \frac{(3^n-1)(4^{n-1}-1)}{6}. \tag{2.10}$$

3. Counting abelian representations

In this section, we show how to compute the Hall invariant $\delta_\Gamma(G)$, in case Γ is a finite abelian group. We start with the well-known computation of the order of $\operatorname{Aut}(\Gamma)$.

For a prime p, denote by Γ_p the p-torsion part of Γ. Then $\Gamma = \bigoplus_{p||\Gamma|} \Gamma_p$, and

$$|\operatorname{Aut}(\Gamma)| = \prod_{p|\,|\Gamma|} |\operatorname{Aut}(\Gamma_p)|\,. \tag{3.1}$$

Let A be a (finite) abelian p-group. Then $A = \mathbb{Z}_{p^{\pi_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\pi_r}}$, for some positive integers $\pi_1 \geq \cdots \geq \pi_r$, and so A determines (and is determined by) a partition $\pi(A) = (\pi_1, \dots, \pi_r)$. Given such a partition π, let $l(\pi) = r$ be its length, and $|\pi| = \sum_{i=1}^r \pi_i$ its weight. Also, let $\langle \pi \rangle = \sum_{i=1}^r (i-1)\pi_i$. Then:

$$|\operatorname{Aut}(\mathbb{Z}_{p^{\pi_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\pi_r}})| = p^{|\pi|+2\langle\pi\rangle} \prod_{k\geq 1} \varphi_{m_k(\pi)}(p^{-1}), \tag{3.2}$$

where $m_k(\pi) = \#\{j \mid \pi_j = k\}$ is the multiplicity of k in λ, and $\varphi_m(t) = \prod_{i=1}^m (1-t^i)$; see Macdonald [30, p. 181][2].

Given a partition λ, let λ' be the partition with $\lambda'_i = \lambda_i - 1$. If τ is another partition, let $\theta_i(\lambda, \tau) = \sum_{j=1}^{l(\tau)} \min(\lambda_i, \tau_j)$, for $1 \leq i \leq l(\lambda)$, and $\theta(\lambda, \tau) = \sum_{i=1}^{l(\lambda)} \theta_i(\lambda, \tau)$. With these notations, we have the following:

Theorem 3.1. *Let G be a finitely-generated group and Γ a finite abelian group. Write $H_1(G) = \mathbb{Z}^n \oplus T$, where T is finite. For each prime p dividing $|\Gamma|$, let $\lambda = \pi(\Gamma_p)$ and $\tau = \pi(T_p)$ be the corresponding partitions. Then:*

$$\delta_\Gamma(G) = \prod_{p|\,|\Gamma|} \frac{p^{(|\lambda|-l(\lambda))n+\theta(\lambda',\tau)} \prod_{i=1}^{l(\lambda)} (p^{n+\theta_i(\lambda,\tau)-\theta_i(\lambda',\tau)} - p^{i-1})}{p^{|\lambda|+2\langle\lambda\rangle} \prod_{k\geq 1} \varphi_{m_k(\lambda)}(p^{-1})}. \tag{3.3}$$

Proof. Since Γ is abelian, every homomorphism $G \to \Gamma$ factors through $H_1(G) = \mathbb{Z}^n \oplus T$. By (2.7), we have $\phi_\Gamma(G) = \prod_{p||\Gamma|} \phi_{\Gamma_p}(\mathbb{Z}^n \oplus T)$. Furthermore, every homomorphism $\mathbb{Z}^n \oplus T \to \Gamma_p$ factors through $\mathbb{Z}^n \oplus T_p$. Hence:

$$\phi_\Gamma(G) = \prod_{p|\,|\Gamma|} \phi_{\Gamma_p}(\mathbb{Z}^n \oplus T_p). \tag{3.4}$$

By Hall's enumeration principle (2.6), we have:

$$\phi_{\Gamma_p}(\mathbb{Z}^n \oplus T_p) = \sum_{H \leq \Gamma_p} \mu(H)\sigma_H(\mathbb{Z}^n \oplus T_p). \tag{3.5}$$

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[2] We are grateful to A. Zelevinsky for pointing this reference to us.

Clearly, $\sigma_H(\mathbb{Z}^n \oplus T_p) = |H|^n\, p^{\theta(\nu,\tau)}$, where $\nu = \pi(H)$. The Möbius function of a subgroup H of Γ_p can be computed from Weisner's formula (2.3), as follows.

Let $\operatorname{Frat}\Gamma_p$ be the Frattini subgroup of Γ_p. Then $\operatorname{Frat}\Gamma_p = (\Gamma_p)^p$, and so the associated partition is λ', where $\lambda = \pi(\Gamma_p)$ and $\lambda'_i = \lambda_i - 1$. If a subgroup H of Γ_p does not contain $\operatorname{Frat}\Gamma_p$, then $\mu(H) = 0$. If H contains $\operatorname{Frat}\Gamma_p$, then the partition $\nu = \pi(H)$ is between λ' and λ, i.e., $\nu_j = \lambda_j - 1$ or λ_j. Order the set $\{j \mid \nu_j = \lambda_j - 1\}$ as $(i_1 \geq \cdots \geq i_d)$. Clearly, $p^d = |\Gamma : H|$, and so $\mu(H) = (-1)^d p^{d(d-1)/2}$.

The number of subgroups $H \leq \Gamma_p$ such that $\operatorname{Frat}\Gamma_p \leq H$ and $\pi(H) = \nu$ equals p^a, where $a = \sum_{k=1}^{d}(i_k - k)$. Set $d_\nu = d$ and $a_\nu = a$. A simple calculation now shows:

$$\begin{aligned}
\phi_{\Gamma_p}(\mathbb{Z}^n \oplus T_p) &= \sum_{\operatorname{Frat}\Gamma_p \leq H \leq \Gamma_p} \mu(H)\sigma_H(\mathbb{Z}^n \oplus T_p) \\
&= \sum_{\lambda' \preceq \nu \preceq \lambda} (-1)^{d_\nu} p^{d_\nu(d_\nu - 1)/2} p^{(|\lambda| - d_\nu)n} p^{\theta(\nu,\tau)} p^{a_\nu} \\
&= p^{(|\lambda| - l(\lambda))n + \theta(\lambda',\tau)} \prod_{i \geq 1} (p^{n + \theta_i(\lambda,\tau) - \theta_i(\lambda',\tau)} - p^{i-1}).
\end{aligned}$$

This, together with formulas (3.1), (3.2), and (3.4), yields (3.3). □

Especially simple is the case when the group G has torsion-free abelianization.

Corollary 3.2. *Let G be a finitely generated group with $H_1(G) = \mathbb{Z}^n$, and Γ a finite abelian group. Write $\Gamma = \prod_{p \mid |\Gamma|} \Gamma_p$, where $\Gamma_p = \mathbb{Z}_{p^{\lambda_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\lambda_r}}$. Then*

$$\delta_\Gamma(G) = \prod_{p \mid |\Gamma|} \frac{p^{|\lambda|(n-1) - 2\langle\lambda\rangle} \varphi_n(p^{-1})}{\varphi_{n-r}(p^{-1}) \prod_{k \geq 1} \varphi_{m_k(\lambda)}(p^{-1})}. \tag{3.6}$$

A formula similar to (3.6) was obtained by Kwak, Chun, and Lee [22, Theorem 3.4]. In particular (writing $\mathbb{Z}_p^s$ for the direct sum of s copies of $\mathbb{Z}_p$):

$$\delta_{\mathbb{Z}_{p^s}}(G) = \tfrac{p^{sn} - p^{(s-1)n}}{p^s - p^{s-1}}, \quad \delta_{\mathbb{Z}_p^s}(G) = \prod_{i=0}^{s-1} \tfrac{p^n - p^i}{p^s - p^i}, \quad \delta_{\mathbb{Z}_p \oplus \mathbb{Z}_{p^s}}(G) = \tfrac{(p^{sn} - p^{(s-1)n})(p^n - p)}{p^{s+1}(p-1)^2}.$$

4. Homology of finite-index subgroups

In this section, we give a formula for computing the first homology (with coefficients in a "sufficiently large" field) of finite-index, normal subgroups of a finitely-presented group.

4.1. **Fox calculus.** Let $G = \langle x_1, \dots, x_\ell \mid r_1, \dots, r_m\rangle$ be a finite presentation for the group G. Let F_ℓ be the free group with generators $x_1, \dots, x_\ell$, and $\phi : F_\ell \to G$ the presenting epimorphism. Let $\mathbb{Z}F_\ell$ be the group-ring of F_ℓ, and $\epsilon : \mathbb{Z}F_\ell \to \mathbb{Z}$ the augmentation map. For each $1 \le j \le \ell$, there is a Fox derivative, $\frac{\partial}{\partial x_j} : \mathbb{Z}F_\ell \to \mathbb{Z}F_\ell$, which is the linear operator defined by the rules $\frac{\partial 1}{\partial x_j} = 0$, $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$, and $\frac{\partial(uv)}{\partial x_j} = \frac{\partial u}{\partial x_j}\epsilon(v) + u\frac{\partial v}{\partial x_j}$.

Let X be the 2-complex associated with the presentation $G = \langle x_1, \dots, x_\ell \mid r_1, \dots, r_m\rangle$. Let $\widetilde{X}$ be the universal cover, and $C_*(\widetilde{X})$ its augmented cellular chain complex. Picking as generators for the chain groups the lifts of the cells of X, the complex $C_*(\widetilde{X})$ becomes identified with

$$(\mathbb{Z}G)^m \xrightarrow{J_G} (\mathbb{Z}G)^\ell \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0, \tag{4.1}$$

where $\partial_1 = \begin{pmatrix} x_1 - 1 & \cdots & x_\ell - 1\end{pmatrix}^\top$, and

$$J_G = \left(\frac{\partial r_i}{\partial x_j}\right)^\phi$$

is the Jacobian matrix of G, obtained by applying the linear extension $\phi : \mathbb{Z}F_\ell \to \mathbb{Z}G$ to the Fox derivatives of the relators.

Clearly, the integral $m \times \ell$ matrix J_G^ϵ is a presentation matrix for $H_1(G)$. More generally, the abelianization of a finite-index subgroup $K \le G$ is given by the following result of Fox.

Theorem 4.2 (Fox [13])**.** *Let G be a finitely-presented group, and $K < G$ a subgroup of index k. Let $J = J_G$ be the Jacobian matrix, $\sigma : G \to \mathrm{Sym}(G/K) \cong S_k$ the coset representation, and $\pi : S_k \to \mathrm{GL}(k, \mathbb{Z})$ the permutation representation. Then $J^{\pi\circ\sigma}$ is a presentation matrix for the abelian group $H_1(K) \oplus \mathbb{Z}^{k-1}$.*

Proof. By Shapiro's Lemma (cf. [4]), $H_1(K, \mathbb{Z})$ is isomorphic to $H_1(G, \mathbb{Z}[G/K])$, the first homology of the chain complex (4.1), tensored over $\mathbb{Z}G$ with the module $\mathbb{Z}[G/K]$. Under the identification $\mathbb{Z}[G/K] = \mathbb{Z}^k$, the boundary map $J \otimes \mathbb{Z}[G/K]$ is the integral $mk \times \ell k$ matrix $J^{\pi\circ\sigma}$. Noting that $\ker \epsilon \otimes \mathbb{Z}[G/K] = \mathbb{Z}^{k-1}$ finishes the proof. □

To compute the abelianization of K by Fox's method, one needs to row-reduce the matrix $J^{\pi\circ\sigma}$. In practical terms, this can be difficult, due to the rather big size of this matrix. If K is a normal subgroup of G, a more efficient method is to first decompose the regular representation of $\Gamma = G/K$ into irreducible representations. Such a method will be described in Theorem 4.6.

4.3. **Representations of finite groups.** Before proceeding, we need some basic facts from the representation theory of finite groups (see [10] as a reference).

Definition 4.4. Let Γ be a finite group, of order $|\Gamma|$. A field $\mathbb{K}$ is *sufficiently large* with respect to Γ if the following two conditions are satisfied:

(i) The characteristic of $\mathbb{K}$ is 0, or coprime to $|\Gamma|$.
(ii) The field $\mathbb{K}$ contains all the e-roots of unity, where e is the exponent of Γ.

Condition (ii) is satisfied if, for example, $\mathbb{K}$ is algebraically closed. If $\Gamma = \mathbb{Z}_p$ is a cyclic group of prime order, and q is a prime different from p, a sufficiently large field is $\mathbb{K} = \mathbb{F}_{q^s}$, the Galois field of order q^s, where $s = \operatorname{ord}_p(q)$ is the least positive integer such that $p \mid (q^s - 1)$.

If condition (i) holds, then the group algebra $\mathbb{K}\Gamma$, viewed as the regular representation of Γ, completely decomposes into irreducible representations (Maschke). If condition (ii) holds, then $\mathbb{K}$ is a splitting field for Γ (Brauer). Thus, if $\mathbb{K}$ sufficiently large, the regular representation $\mathbb{K}\Gamma$ decomposes into (absolutely) irreducible representations:

$$\mathbb{K}\Gamma = \bigoplus_{\rho \in Z} \bigoplus_{n_\rho} W_\rho, \tag{4.2}$$

where $Z = \operatorname{Irrep}(\Gamma, \mathbb{K})$ is the set of isomorphism classes of irreducible $\mathbb{K}$ representations of Γ, and n_ρ is the dimension of the representation $\rho : \Gamma \to \operatorname{GL}(W_\rho)$. In particular, $|\Gamma| = \sum_{\rho \in Z} n_\rho^2$.

4.5. **Mod q Betti numbers.** Let $b_1(G) = \operatorname{rank} H_1(G) = \dim_{\mathbb{Q}} H_1(G; \mathbb{Q})$ be the first Betti number of G. We shall write $b_1^{(0)} = b_1$. For a prime q, set

$$b_1^{(q)}(G) = \dim_{\mathbb{F}_q} H_1(G; \mathbb{F}_q).$$

Since homology commutes with direct sums, we have $b_1^{(q)}(G) = \dim_{\mathbb{K}} H_1(G; \mathbb{K})$, for any field $\mathbb{K}$ of characteristic q. We will call $b_1^{(q)}(G)$ the *mod q (first) Betti number of G*.

Theorem 4.6. *Let G be a finitely-presented group. Let $\lambda : G \twoheadrightarrow \Gamma$ be a representation of G onto a finite group Γ. Let $\mathbb{K}$ be a field, sufficiently large with respect to Γ. Then, if $K_\lambda = \ker \lambda$ and $q = \operatorname{char} \mathbb{K}$:*

$$b_1^{(q)}(K_\lambda) = b_1^{(q)}(G) + \sum_{\rho \neq \mathbf{1}} n_\rho (\operatorname{corank} J^{\rho \circ \lambda} - n_\rho), \tag{4.3}$$

where $J^{\rho\circ\lambda}$ is the Jacobian matrix of G, followed by the representation $\rho \circ \lambda : G \to \operatorname{GL}(n_\rho, \mathbb{K})$, and ρ runs through all non-trivial, irreducible $\mathbb{K}$-representations of Γ.

Proof. Let $G = \langle x_1, \dots, x_\ell \mid r_1, \dots, r_m \rangle$ be a finite presentation. Since $\operatorname{char} \mathbb{K} = q$, we have $b_1^{(q)}(K_\lambda) = \dim_{\mathbb{K}} H_1(K_\lambda; \mathbb{K})$. As in the proof of Fox's Theorem 4.2,

$H_1(K_\lambda, \mathbb{K})$ is the first homology of the chain complex

$$(\mathbb{K}\Gamma)^m \xrightarrow{J^\lambda} (\mathbb{K}\Gamma)^\ell \xrightarrow{\partial_1^\lambda} \mathbb{K}\Gamma \xrightarrow{\epsilon} \mathbb{K} \to 0, \tag{4.4}$$

obtained by tensoring (4.1) with $\mathbb{K}\Gamma$ (viewed as a G-module via the representation $\lambda : G \to \Gamma$). In other words, $b_1^{(q)}(K_\lambda) = \dim_{\mathbb{K}} \ker \partial_1^\lambda - \dim_{\mathbb{K}} \operatorname{im} J^\lambda$.

Let $Z = \operatorname{Irrep}(\Gamma, \mathbb{K})$. In view of (4.2), the chain complex (4.4) decomposes as:

$$\bigoplus_{\rho\in Z} \bigoplus_{n_\rho} W_\rho^m \xrightarrow{\oplus_\rho \oplus_{n_\rho} J^{\rho\circ\lambda}} \bigoplus_{\rho\in Z} \bigoplus_{n_\rho} W_\rho^\ell \xrightarrow{\oplus_\rho \oplus_{n_\rho} \partial_1^{\rho\circ\lambda}} \bigoplus_{\rho\in Z} \bigoplus_{n_\rho} W_\rho \xrightarrow{\epsilon} \mathbb{K} \to 0.$$

The twisted Jacobian matrices $J^{\rho\circ\lambda}$ are of size $mn_\rho \times \ell n_\rho$, and have entries in $\mathbb{K}$. By Theorem 4.2, $\operatorname{rank} J^{\mathbf{1}\circ\lambda} = \ell - b_1^{(q)}(G)$. Hence, $\dim_{\mathbb{K}} \operatorname{im} J^\lambda = \ell - b_1^{(q)}(G) + \sum_{\rho\neq\mathbf{1}} n_\rho \operatorname{rank} J^{\rho\circ\lambda}$.

Now notice that ∂_1^λ has rank equal to $\dim_{\mathbb{K}} \ker \epsilon = |\Gamma| - 1$. Hence, $\dim_{\mathbb{K}} \ker \partial_1^\lambda = \ell\,|\Gamma| - (|\Gamma| - 1)$. Therefore,

$$b_1^{(q)}(K_\lambda) = \ell\,|\Gamma| - |\Gamma| + 1 - \big(\ell - b_1^{(q)}(G) + \sum_{\rho\neq\mathbf{1}} n_\rho \operatorname{rank} J^{\rho\circ\lambda}\big).$$

Since $|\Gamma| = 1 + \sum_{\rho\neq\mathbf{1}} n_\rho^2$, we get formula (4.3). □

Notice that $\operatorname{corank} J^{\rho\circ\lambda} \geq \operatorname{rank} \partial_1^{\rho\circ\lambda} = n_\rho$. Hence, each term in the sum (4.3) is non-negative, and so

$$b_1^{(q)}(K_\lambda) \geq b_1^{(q)}(G). \tag{4.5}$$

In view of this inequality, we are led to the following definition.

Definition 4.7. Let G be a finitely-presented group, and let Γ be a finite group. For $q = 0$, or q a prime not dividing $|\Gamma|$, and d a non-negative integer, put

$$\beta_{\Gamma,d}^{(q)}(G) := \#\big\{K \lhd G \mid G/K \cong \Gamma \quad \text{and} \quad b_1^{(q)}(K) = b_1^{(q)}(G) + d\,\big\}.$$

In other words, $\beta_{\Gamma,d}^{(q)}(G)$ counts those normal subgroups of G, with factor group Γ, for which the mod q first Betti number jumps by d, when compared to that of G. Notice that:

$$\sum_{d\geq 0} \beta_{\Gamma,d}^{(q)}(G) = \delta_\Gamma(G). \tag{4.6}$$

4.8. **Homology of finite abelian covers.** We may further refine Theorem 4.6 in the case when the group Γ is abelian. We start with an immediate corollary.

If Γ is abelian, then all its irreducible representations over a sufficiently large field $\mathbb{K}$ are 1-dimensional. Hence, taking $\mathbb{K} = \mathbb{C}$ in the above theorem, we obtain:

Corollary 4.9 (Libgober [23], Sakuma [39], Hironaka [21])**.** *Let $\lambda : G \twoheadrightarrow \Gamma$ be a representation of a finitely-presented group G onto a finite abelian group Γ. If $K_\lambda = \ker \lambda$, then:*

$$b_1(K_\lambda) = b_1(G) + \sum_{\rho \neq \mathbf{1}} (\operatorname{corank} J^{\rho\circ\lambda} - 1), \tag{4.7}$$

where ρ runs through all non-trivial, irreducible, complex representations of Γ.

For a representation $\rho : \Gamma \to \mathbb{K}^*$, let $\langle\rho\rangle$ be the (cyclic) subgroup of $\operatorname{Hom}(\Gamma, \mathbb{K}^*)$ generated by ρ, and set $m_\rho = |\langle\rho\rangle|$. Let $Z^\wedge = \operatorname{Irrep}^\wedge(\Gamma, \mathbb{K})$ be a set of representatives for the non-trivial, irreducible $\mathbb{K}$-representations of Γ, under the equivalence relation $\rho_1 \sim \rho_2 \iff \langle\rho_1\rangle = \langle\rho_2\rangle$.

Theorem 4.10. *Let $\lambda : G \twoheadrightarrow \Gamma$ be a representation of a finitely-presented group G onto a finite abelian group Γ. Let $\mathbb{K}$ be a sufficiently large field, of characteristic q. Then:*

$$b_1^{(q)}(K_\lambda) = b_1^{(q)}(G) + \sum_{\rho \in Z^\wedge} m_\rho (\operatorname{corank} J^{\rho\circ\lambda} - 1). \tag{4.8}$$

Proof. Assume $\rho_1 \sim \rho_2$. Let C be the cyclic group generated by ρ_1. Then there is an automorphism $\psi : C \to C$ such that $\psi(\rho_1) = \rho_2$. The linear extension $\psi : \mathbb{K}C \to \mathbb{K}C$ is an isomorphism, taking $J^{\rho_1\circ\lambda}$ to $J^{\rho_2\circ\lambda}$. Consequently, the $\mathbb{K}$-modules presented by these two matrices are isomorphic. Hence, $\operatorname{corank} J^{\rho_1\circ\lambda} = \operatorname{corank} J^{\rho_2\circ\lambda}$, and so the contributions of ρ_1 and ρ_2 to the sum (4.3) are equal. $\square$

The above theorem permits us to derive bounds and congruences on the mod q Betti numbers of normal subgroups $K \triangleleft G$ with $\Gamma = G/K$ finite abelian, provided $q \nmid |\Gamma|$.

Corollary 4.11. *Let G be a finitely-presented group, and K a normal subgroup with G/K finite abelian. Suppose $q = 0$, or q is a prime not dividing $k = |G/K|$.*

(1) *Let $\ell(G)$ be the minimal number of generators in a finite presentation for G. Then:*
$$b_1^{(q)}(G) \leq b_1^{(q)}(K) \leq b_1^{(q)}(G) + (k-1)(\ell(G) - 1).$$
(2) *Let $p_1, \dots, p_r$ be the prime factors of k, and set $D = \gcd(p_1 - 1, \dots, p_r - 1)$. Then:*
$$b_1^{(q)}(K) \equiv b_1^{(q)}(G) \mod D.$$

Proof. (1) The first inequality was already noted in (4.5). To prove the second inequality, pick a finite presentation of G with $\ell = \ell(G)$ generators, so that the Jacobian matrix $J = J_G$ has ℓ columns. For each representation $\rho \in Z^\wedge$, the matrix $J^{\rho\circ\lambda}$ also has ℓ columns, since ρ is one-dimensional. Hence, $\operatorname{corank} J^{\rho\circ\lambda} \leq \ell$, for all $\rho \in Z^\wedge$.

(2) Each m_ρ is of the form $\phi(r)$, for some integer $r > 1$ dividing k. Hence, $D \mid m_\rho$, for all $\rho \in Z^\wedge$. $\square$

5. Alexander matrices and characteristic varieties

In this section, we introduce the characteristic varieties of a finitely-presented group G, over an arbitrary field $\mathbb{K}$.

5.1. **Alexander matrix.** Let $G = \langle x_1, \dots, x_\ell \mid r_1, \dots, r_m \rangle$ be a finitely presented group. Let $\phi : F_\ell \twoheadrightarrow G$ be the presenting homomorphism, and $\alpha : G \twoheadrightarrow H_1(G)$ the abelianization map. Fix an isomorphism $\chi : H_1(G) \xrightarrow{\cong} \mathbb{Z}^n \oplus \bigoplus_{i=1}^h \mathbb{Z}_{e_i}^{n_i}$, where e_i are distinct elementary divisors. This identifies the group-ring $\mathbb{Z}H_1(G)$ with the ring

$$\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}, s_{1,1}, \dots, s_{1,n_1}, \dots, s_{h,1}, \dots, s_{h,n_h}]/(s_{i,j}^{e_i} - 1). \tag{5.1}$$

The *Alexander matrix* of G is the $\ell \times m$ matrix with entries in Λ given by

$$A_G = J_G^{\chi\circ\alpha}, \tag{5.2}$$

where recall $J_G = \left(\frac{\partial r_i}{\partial x_j}\right)^\phi$ is the Fox Jacobian matrix associated to the given presentation of G. The *d-th Alexander ideal*, $E_d(A_G)$, is the ideal of Λ generated by the codimension d minors of A_G. As is well-known, this ideal does not depend on the choice of presentation for G (but it does depend on the choice of isomorphism χ).

5.2. **Characteristic varieties.** Let $\mathbb{K}$ be a field, and let $\mathbb{K}^*$ be its multiplicative group of units. For N a positive integer, let $\Omega_{N,\mathbb{K}}$ be the set of roots of unity of order N in $\mathbb{K}$.

Let $\mathrm{Hom}(G, \mathbb{K}^*)$ be the group of $\mathbb{K}$-valued characters of G. The isomorphism $\chi : H_1(G) \to \mathbb{Z}^n \oplus \mathbb{Z}_{e_1}^{n_1} \oplus \cdots \oplus \mathbb{Z}_{e_h}^{n_h}$ identifies the character variety $\mathrm{Hom}(G, \mathbb{K}^*)$ with the product of affine algebraic tori

$$\mathbb{T} = (\mathbb{K}^*)^n \times (\Omega_{e_1,\mathbb{K}})^{n_1} \times \cdots \times (\Omega_{e_h,\mathbb{K}})^{n_h}, \tag{5.3}$$

viewed as a subset of the torus $(\mathbb{K}^*)^{n+n_1+\cdots+n_h}$.

Definition 5.3. The *d-th characteristic variety* of the group G (over the field $\mathbb{K}$) is the subvariety $V_d(G, \mathbb{K})$ of the algebraic torus $\mathbb{T} = \mathrm{Hom}(G, \mathbb{K}^*)$, consisting of characters $\mathbf{t} : G \to \mathbb{K}^*$ such that $f(\mathbf{t}) = 0$, for all $f \in E_d(A_G) \otimes \mathbb{K}$.

In other words, $V_d(G, \mathbb{K})$ is the d-th determinantal variety of $A_G \otimes \mathbb{K}$. As such, $V_d(G, \mathbb{K})$ is also defined by the annihilator of the d-th exterior power of the Alexander module, $\operatorname{coker} A_G \otimes \mathbb{K}$, see [11, pp. 511–513]. If $d < \ell(G)$, this module has the same support as the Alexander invariant, $H_1(G', \mathbb{K})$, where G' is the commutator subgroup of G.

The characteristic varieties of G form a descending tower, $\mathbb{T} = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{\ell(G)-1} \supseteq V_{\ell(G)} = \emptyset$. This tower depends only on the isomorphism type of G, up to a monomial change of basis in the algebraic torus $(\mathbb{K}^*)^{n+n_1+\cdots+n_h}$.

Remark 5.4. The characteristic varieties $V_d(G,\mathbb{K})$ may be interpreted as the jumping loci for the cohomology of G with coefficients in rank 1 local systems over $\mathbb{K}$. More precisely, let

$$\Sigma_d(G,\mathbb{K}) = \{\mathbf{t} \in \mathbb{T} \mid \dim_{\mathbb{K}} H^1(G,\mathbb{K}_{\mathbf{t}}) \geq d\}, \tag{5.4}$$

where $\mathbb{K}_{\mathbf{t}}$ is the G-module $\mathbb{K}$ with action given by the representation $\mathbf{t} : G \to \mathbb{K}^*$. Then, $V_d(G,\mathbb{K}) \setminus \{\mathbf{1}\} = \Sigma_d(G,\mathbb{K}) \setminus \{\mathbf{1}\}$.

For $\mathbb{K} = \mathbb{C}$, this was proved by Hironaka [21] (see also Libgober [24] and Cogolludo [7]). The proof given in [24, 7] can be adapted to work for an arbitrary field $\mathbb{K}$. Recall that $V_d(G,\mathbb{K})$ is defined by $\operatorname{ann}\big(\bigwedge^d(H_1(G',\mathbb{K}))\big)$. Thus, $\mathbf{t} \in V_d(G,\mathbb{K}) \iff \dim_{\mathbb{K}} H_1(G,\mathbb{K}_{\mathbf{t}}) \geq d$ (here we need $\mathbf{t} \neq \mathbf{1}$). But $H_1(G,\mathbb{K}_{\mathbf{t}}) \cong H^1(G,\mathbb{K}_{\mathbf{t}^{-1}})$, see [3, p. 341], and we are done.

Remark 5.5. Closely related are the *resonance varieties* of the group G (over the field $\mathbb{K}$), defined as $R_d(G,\mathbb{K}) = \{\lambda \in H^1(G,\mathbb{K}) \mid \dim_{\mathbb{K}} H^1(H^*(G,\mathbb{K}),\cdot\lambda) \geq d\}$, see [12, 26, 33]. If all the relators of G are commutators ($r_j \in [F_\ell,F_\ell]$, $\forall j$), then $R_d(G,\mathbb{K})$ is the d-th determinantal variety of the *linearized* Alexander matrix of G, see [33]. Moreover, as shown by Libgober [25], the tangent cone at $\mathbf{1}$ to $V_d(G,\mathbb{C})$ is included in $R_d(G,\mathbb{C})$.

5.6. **Depth of characters.** Let $\mathbf{t} : G \to \mathbb{K}^*$ be a character. Since $\mathbb{K}^*$ is an abelian group, $\mathbf{t}$ factors through the abelianization $\alpha : G \to H_1(G)$. Let $A^{\mathbf{t}} : \mathbb{K}^m \to \mathbb{K}^\ell$ be the matrix obtained from A by evaluating at $\mathbf{t}$. (Under the isomorphism $\chi : \mathbb{K}H_1(G) \to \Lambda \otimes \mathbb{K}$, the twisted Jacobian matrix $J^{\mathbf{t}}$ corresponds to the twisted Alexander matrix $A^{\mathbf{t}}$.) We then have:

$$\mathbf{t} \in V_d(G,\mathbb{K}) \iff \operatorname{rank}_{\mathbb{K}} A^{\mathbf{t}} \leq \ell - d - 1. \tag{5.5}$$

Definition 5.7. Let G be a finitely-presented group, and $\mathbb{K}$ a field. The *depth* of a character $\mathbf{t} : G \to \mathbb{K}^*$ (relative to the stratification of $\operatorname{Hom}(G,\mathbb{K}^*)$ by the characteristic varieties) is:

$$d_{\mathbb{K}}(\mathbf{t}) = \max\{d \mid \mathbf{t} \in V_d(G,\mathbb{K})\}.$$

Note that $0 \leq d_{\mathbb{K}}(\mathbf{t}) \leq \ell(G) - 1$. Thus, we can sharpen the upper bound from Corollary 4.11(1), as follows. Let $K \triangleleft G$ be a normal subgroup, with G/K abelian of order k, and choose $\mathbb{K}$ to be sufficiently large with respect to G/K. Then:

$$b_1^{(q)}(K) \leq b_1^{(q)}(G) + (k-1)d_{\mathbb{K}}(G), \tag{5.6}$$

where $d_{\mathbb{K}}(G) = \sup\{d_{\mathbb{K}}(\mathbf{t}) \mid \mathbf{1} \neq \mathbf{t} \in \operatorname{Hom}(G,\mathbb{K}^*)\}$.

6. Torsion points and Betti numbers

Given a normal subgroup of $K \triangleleft G$ with finite cyclic quotient, we interpret the first homology of K with coefficients in a sufficiently large field $\mathbb{K}$, in terms of the stratification of the character variety $\operatorname{Hom}(G, \mathbb{K}^*)$ by the characteristic varieties of G.

6.1. **Homology of normal subgroups with cyclic quotient.** Let $\Gamma = \mathbb{Z}_N$ be a finite cyclic group. Let $\mathbb{K}$ be a field. Assume that $\mathbb{K}$ is sufficiently large with respect to $\mathbb{Z}_N$. Then $\mathbb{K}$ contains all the N-th roots of unity, and so there is a monomorphism $\iota : \mathbb{Z}_N \hookrightarrow \mathbb{K}^*$, sending a generator of $\mathbb{Z}_N$ to a primitive N-th root of unity in $\mathbb{K}^*$. Finally, for $j \geq 0$, let $\psi_j : \mathbb{K} \to \mathbb{K}$ be the map $\psi_j(x) = x^j$.

Theorem 6.2. *Let $\lambda : G \twoheadrightarrow \mathbb{Z}_N$ be a surjective homomorphism. Let $\mathbb{K}$ be a field, sufficiently large with respect to $\mathbb{Z}_N$. Set $K_\lambda = \ker(\lambda)$, and $q = \operatorname{char} \mathbb{K}$. Then*

$$b_1^{(q)}(K_\lambda) = b_1^{(q)}(G) + \sum_{1 \neq k | N} \phi(k) d_{\mathbb{K}}(\lambda^{N/k}), \tag{6.1}$$

where $\lambda^{N/k} = \psi_{N/k} \circ \iota \circ \lambda$.

Proof. By Theorem 4.10, we have

$$b_1^{(q)}(K_\lambda) = b_1^{(q)}(G) + \sum_{\rho \in Z^\wedge} m_\rho (\operatorname{corank} J^{\rho\circ\lambda} - 1). \tag{6.2}$$

Since $\mathbb{K}$ is sufficiently large, $\operatorname{Hom}(\mathbb{Z}_N, \mathbb{K}^*) \cong \mathbb{Z}_N$. Let $\operatorname{ord}(\rho)$ be the order of a non-trivial representation $\rho : \mathbb{Z}_N \to \mathbb{K}^*$. It is readily seen that $\rho \sim \rho' \iff \operatorname{ord}(\rho) = \operatorname{ord}(\rho')$. Thus, the assignment $\rho \mapsto \operatorname{ord}(\rho)$ establishes a bijection between $Z^\wedge = \operatorname{Irrep}^\wedge(\mathbb{Z}_N, \mathbb{K})$ and the set of non-unit divisors of N. Moreover, $m_\rho = \phi(k)$, where $k = \operatorname{ord}(\rho)$.

Now consider the representation $\rho = \psi_{N/k} \circ \iota$. Clearly, the order of ρ is k. Let $\lambda^\vee : \operatorname{Hom}(\mathbb{Z}_N, \mathbb{K}^*) \to \operatorname{Hom}(G, \mathbb{K}^*)$ be the dual homomorphism. We then have $\lambda^\vee(\rho) = \lambda^{N/k}$. Furthermore, by (5.5), we have

$$\lambda^\vee(\rho) \in V_d(G, \mathbb{K}) \iff \operatorname{corank}_{\mathbb{K}} A^{\rho\circ\lambda} \geq d + 1. \tag{6.3}$$

The conclusion follows at once. $\square$

Corollary 6.3. *Let $K \triangleleft G$ be a normal subgroup of prime index p. Write $K = \ker(\lambda : G \to \mathbb{Z}_p)$. Let $q = 0$, or q a prime, $q \neq p$. Let $\mathbb{K}$ be a field of characteristic q which contains all the p-roots of unity—for example, $\mathbb{K} = \mathbb{C}$, or $\mathbb{K} = \mathbb{F}_{q^s}$, where $s = \operatorname{ord}_p(q)$. Then:*

$$b_1^{(q)}(K) = b_1^{(q)}(G) + (p - 1) d_{\mathbb{K}}(\lambda). \tag{6.4}$$

6.4. **Distribution of mod q Betti numbers.** In view of the above corollary, it makes sense to define

$$\beta^{(q)}_{p,d}(G) := \beta^{(q)}_{\mathbb{Z}_p,(p-1)d}(G) \tag{6.5}$$

In other words, $\beta^{(q)}_{p,d}(G)$ counts those index p, normal subgroups of G for which the mod q first Betti number jumps by $(p-1)d$, when compared to that of G. By (4.6), we have $\sum_{d\geq 0}\beta^{(q)}_{p,d}(G)=\delta_{\mathbb{Z}_p}(G)$. Hence, by Theorem 3.1:

$$\sum_{d\geq 0}\beta^{(q)}_{p,d}(G)=\frac{p^n-1}{p-1},\quad \text{where } n=b_1^{(p)}(G). \tag{6.6}$$

For simplicity, we shall write sometimes $\beta^{(q)}_p=\big(\beta^{(q)}_{p,1},\dots,\beta^{(q)}_{p,k}\big)$, if $\beta^{(q)}_{p,d}=0$, for $d>k$. Also, we will abbreviate $\beta_{p,d}=\beta^{(0)}_{p,d}$. Note that $\beta^{(q)}_{p,0}$ is determined from (6.6) by the sequence $\beta^{(q)}_p$, and the mod p first Betti number of G.

Let

$$\operatorname{Tors}_{p,d}(G,\mathbb{K})=\{\mathbf{t}\in\operatorname{Hom}(G,\mathbb{K}^*)\mid \mathbf{t}^p=\mathbf{1} \text{ and } \mathbf{t}\neq\mathbf{1}\}\cap V_d(G,\mathbb{K}) \tag{6.7}$$

be the set of characters on $V_d(G,\mathbb{K})$ of order exactly equal to p. As a direct consequence of Corollary 6.3, we obtain:

Theorem 6.5. *Let G be a finitely-presented group, p a prime, and $q=0$, or q a prime, distinct from p. If $\mathbb{K}$ is a field of characteristic q containing all p-roots of unity, then:*

$$\beta^{(q)}_{p,d}(G)=\frac{|\operatorname{Tors}_{p,d}(G,\mathbb{K})\setminus\operatorname{Tors}_{p,d+1}(G,\mathbb{K})|}{p-1}.$$

In particular, $\beta_{p,d}(G)=\frac{1}{p-1}\left|\operatorname{Tors}_{p,d}(G,\mathbb{C})\setminus\operatorname{Tors}_{p,d+1}(G,\mathbb{C})\right|$. Also, if $q>0$, then $\beta^{(q)}_{p,d}(G)=\frac{1}{p-1}\left|\operatorname{Tors}_{p,d}(G,\mathbb{F}_{q^s})\setminus\operatorname{Tors}_{p,d+1}(G,\mathbb{F}_{q^s})\right|$, where $s=\operatorname{ord}_p(q)$.

Remark 6.6. This result does not say anything about the distribution of mod p Betti numbers of index p subgroups. Even so, there is a class of groups for which an analogous formula holds for $q=p$, with the characteristic varieties replaced by the resonance varieties (over the field $\mathbb{F}_p$). Indeed, let $G=\langle x_1,\dots,x_n\mid r_1,\dots,r_m\rangle$ be a commutator-relators group, with $H_2(G)$ torsion-free, and let $Q=G/[G,[G,G]]$ be its second nilpotent quotient. Set $\nu_{p,d}(Q)=\#\big\{K\lhd Q\mid |Q:K|=p \text{ and } b_1^{(p)}(K)=n+d\,\big\}$. Then, according to [33, Theorem 4.19]:

$$\nu_{p,d}(Q)=\tfrac{1}{p-1}\left|R_d(Q,\mathbb{F}_p)\setminus R_{d+1}(Q,\mathbb{F}_p)\right|. \tag{6.8}$$

6.7. **Computations of β-invariants.** We conclude this section with some sample computations of the invariants $\beta^{(q)}_{p,d}(G)$, for some familiar finitely-presented groups G.

Example 6.8. Let $G = F_n$ be the free group of rank n. Evidently, $V_0(G,\mathbb{K}) = \cdots = V_{n-1}(G,\mathbb{K}) = (\mathbb{K}^*)^n$, and $V_n(G,\mathbb{K}) = \{\mathbf{1}\}$, for all $\mathbb{K}$. Hence, for all q:

$$\beta^{(q)}_{p,n-1}(G) = \frac{p^n-1}{p-1}, \quad \text{and} \quad \beta^{(q)}_{p,d}(G) = 0, \text{ for } d \neq n-1.$$

Example 6.9. Let $G = F_m \times F_n$ ($m \geq n$) be the product of two free groups. Then:

$$V_d(G,\mathbb{K}) = \begin{cases} (\mathbb{K}^*)^{m+n} & \text{if } d = 0, \\ (\mathbb{K}^*)^n \cup (\mathbb{K}^*)^m & \text{if } 0 < d < n \\ (\mathbb{K}^*)^m & \text{if } n \leq d < m, \\ \{\mathbf{1}\} & \text{if } d = m. \end{cases}$$

where $(\mathbb{K}^*)^n = \{t_1 = \cdots = t_m = 1\}$ and $(\mathbb{K}^*)^m = \{t_{m+1} = \cdots = t_{m+n} = 1\}$ (see [9]). Hence: $\beta^{(q)}_{p,0}(G) = \frac{(p^m-1)(p^n-1)}{p-1}$, $\beta^{(q)}_{p,n-1}(G) = \frac{p^n-1}{p-1}$, $\beta^{(q)}_{p,m-1}(G) = \frac{p^m-1}{p-1}$, and $\beta^{(q)}_{p,d}(G) = 0$, otherwise.

More generally, consider the product $G = F_{n_1} \times \cdots \times F_{n_k}$, with $n_1 \geq \cdots \geq n_k$. Write $\mathbf{n} = (n_1, \dots, n_k)$, and recall that $|\mathbf{n}| = \sum_{i=1}^k n_i$ and $m_d(\mathbf{n}) = \#\{j \mid n_j = d\}$. We then have:

$$\beta^{(q)}_{p,0}(G) = \frac{p^{|\mathbf{n}|} - 1 - \sum_{i=1}^k (p^{n_i} - 1)}{p-1}, \qquad \beta^{(q)}_{p,d-1}(G) = m_d(\mathbf{n}) \frac{p^d-1}{p-1}, \quad \text{for } d > 1.$$

Example 6.10. Let $G = \pi_1(\#_g T^2)$ be the fundamental group of a closed, orientable surface of genus $g \geq 1$, with presentation $G = \langle x_1, \dots, x_{2g} \mid [x_1,x_2]\cdots[x_{2g-1},x_{2g}] = 1\rangle$. Then: $V_d(G,\mathbb{K}) = (\mathbb{K}^*)^{2g}$, for $d < 2g-1$, and $V_{2g-1}(G,\mathbb{K}) = \{\mathbf{1}\}$ (see [21]). Hence:

$$\beta^{(q)}_{p,2g-2}(G) = \frac{p^{2g}-1}{p-1}, \text{ and } \beta^{(q)}_{p,d}(G) = 0, \text{ for } d \neq 2g-2.$$

Example 6.11. Let $G = \pi_1(\#_n \mathbb{RP}^2)$ be the fundamental group of a closed, non-orientable surface of genus $n \geq 1$, with presentation $G = \langle x_1, \dots, x_n \mid x_1^2 \cdots x_n^2 = 1\rangle$. The isomorphism $\chi : H_1(G) \to \mathbb{Z}^{n-1} \oplus \mathbb{Z}_2$, given by $\chi(x_i) = t_i$ for $i < n$ and $\chi(x_1 \cdots x_n) = s$, identifies $\mathbb{Z}H_1(G)$ with $\mathbb{Z}[t_1^{\pm 1}, \dots, t_{n-1}^{\pm 1}, s]/(s^2-1)$. The Alexander matrix A_G is

$$\begin{pmatrix} 1+t_1 & t_1^2(1+t_2) & \cdots & t_1^2\cdots t_{n-2}^2(1+t_{n-1}) & t_1^2\cdots t_{n-1}^2(1+t_1^{-1}\cdots t_{n-1}^{-1}s) \end{pmatrix}.$$

If $\operatorname{char}\mathbb{K} \neq 2$, the character variety $\operatorname{Hom}(G,\mathbb{K}^*)$ is isomorphic to $\mathbb{T} = (\mathbb{K}^*)^{n-1} \times \{\pm 1\}$, and the characteristic varieties are: $V_0 = \cdots = V_{n-2} = \mathbb{T}$, and $V_{n-1} =$

$\{(-1,\dots,-1,(-1)^n)\}$. If $\operatorname{char}\mathbb{K}=2$, then $V_0=\cdots=V_{n-2}=(\mathbb{K}^*)^{n-1}$, and $V_{n-1}=\{\mathbf{1}\}$. Hence:

$$\beta^{(q)}_{2,n-2}(G)=2^n-2,\quad \beta^{(q)}_{2,n-1}(G)=1,\quad \beta^{(q)}_{p,n-2}(G)=\frac{p^{n-1}-1}{p-1},$$

and $\beta^{(q)}_{p,d}(G)=0$, otherwise.

7. Counting metabelian representations

We now return to the Hall invariants of a finitely-presented group G. We show how to compute $\delta_\Gamma(G)$, for the split metabelian groups $\Gamma=\mathbb{Z}_q^s\rtimes\mathbb{Z}_p$, in terms of torsion points on the characteristic varieties of G, over the Galois field $\mathbb{F}_{q^s}$.

7.1. **A class of metabelian groups.** For two distinct primes p and q, we define the metabelian group M_{p,q^s} to be the (non-trivial) split extension

$$M_{p,q^s}=\mathbb{Z}_q^s\rtimes_\sigma\mathbb{Z}_p=\langle a_1,\dots,a_s,b\mid a_i^q=[a_i,a_j]=b^p=1,b^{-1}a_ib=\sigma(a_i)\rangle, \tag{7.1}$$

where $s=\operatorname{ord}_p(q)$ is the order of q mod p in $\mathbb{Z}_p^*$, and σ is an automorphism of $\mathbb{Z}_q^s$, of order exactly p.

Note that σ must act trivially on any proper, invariant subgroup of $\mathbb{Z}_q^s$, and so, all proper subgroups of M_{p,q^s} are abelian. Implicit in the definition is the assertion that such automorphism σ exists, and that the isomorphism type of M_{p,q^s} does not depend on its choice. This is proved in the following lemma, which also gives the order of the automorphism group of M_{p,q^s}.

Lemma 7.2. *Let p and q be distinct primes, and let $s=\operatorname{ord}_p(q)$. Then:*

(1) *There exists an automorphism $\sigma\in\operatorname{Aut}(\mathbb{Z}_q^s)$ of order p.*
(2) *If ψ is another automorphism of $\mathbb{Z}_q^s$ of order p, then $\mathbb{Z}_q^s\rtimes_\sigma\mathbb{Z}_p\cong\mathbb{Z}_q^s\rtimes_\psi\mathbb{Z}_p$.*
(3) $\left|\operatorname{Aut}(\mathbb{Z}_q^s\rtimes_\sigma\mathbb{Z}_p)\right|=sq^s(q^s-1)$.

Proof. (1) The cyclotomic polynomial $Q_p=t^{p-1}+\cdots+t+1$ factors over the field $\mathbb{F}_q$ into $(p-1)/s$ distinct, monic, irreducible polynomials of degree s. If f is any one of those factors, then the field $\mathbb{F}_{q^s}$ is isomorphic to $\mathbb{F}_q[t]/(f)$. Let σ be the automorphism of $\mathbb{F}_q[t]/(f)$ induced by multiplication by t in $\mathbb{F}_q[t]$. Clearly, σ has order p.

Note the following: If we view $\mathbb{Z}_q^s$ as the $\mathbb{F}_q$-vector space with basis $\{1,t,\dots,t^{s-1}\}$, then $\sigma\in\operatorname{Aut}(\mathbb{Z}_q^s)\cong\operatorname{GL}(s,q)$ may be identified with the companion matrix of f, and so $f=f_\sigma$, the characteristic polynomial of σ. Alternatively, if we view $\mathbb{Z}_q^s$ as the additive group of the field $\mathbb{F}_{q^s}=\mathbb{F}_q(\xi)$, where ξ is a primitive p-th root of unity, then $\sigma=\cdot\xi\in\operatorname{Aut}(\mathbb{F}_q(\xi))$.

(2) Notice that $\mathbb{Z}_p^s \rtimes_\sigma \mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p^s \rtimes_{\sigma^l} \mathbb{Z}_p$, for any $0 < l < p$: The mapping $a_i \mapsto a_i$, $b \mapsto b^k$, where $k = l^{-1}$ in the multiplicative group $\mathbb{Z}_p^*$, provides such an isomorphism.

Now let ψ be an arbitrary matrix of order p in $\mathrm{GL}(s,q)$. The characteristic polynomial of ψ must be one of the $(p-1)/s$ irreducible factors of Q_p. All such factors are the characteristic polynomials of some power of σ. Thus, $f_\psi = f_{\sigma^l}$, for some $0 < l < p$. An exercise in linear algebra shows that there is a matrix $\phi \in \mathrm{GL}(s,q)$ such that $\psi = \phi\sigma^l\phi^{-1}$. Hence, $\mathbb{Z}_q^s \rtimes_\psi \mathbb{Z}_p \cong \mathbb{Z}_q^s \rtimes_{\sigma^l} \mathbb{Z}_p$.

(3) Let $\Phi \in \mathrm{Aut}(\mathbb{Z}_q^s \rtimes_\sigma \mathbb{Z}_p)$. Every element in the semi-direct product $\mathbb{Z}_q^s \rtimes_\sigma \mathbb{Z}_p$ has unique normal form ub^k, for some $u \in \mathbb{Z}_q^s$ and $0 \le k \le p-1$. Write $\Phi(b) = vb^l$. Straightforward computations show that Φ leaves the subgroup $\mathbb{Z}_q^s$ invariant, and that the restriction $\phi = \Phi|_{\mathbb{Z}_q^s}$ satisfies $\phi\sigma\phi^{-1} = \sigma^l$. The number of solutions $\phi \in \mathrm{GL}(s,q)$ of this equation is $q^s - 1$.

The count of automorphisms of M_{p,q^s} then follows from the following claim: There are precisely s values $1 \le l \le p-1$ for which $f_{\sigma^l} = f_\sigma$. To prove the claim, notice that $\prod_{l=1}^{p-1} f_{\sigma^l} = Q_p^s$ (both polynomials factor completely over $\mathbb{F}_{q^s}$ into linear factors, and the factorizations coincide). But, over $\mathbb{F}_q$, the polynomial Q_p has $(p-1)/s$ distinct irreducible factors, all with multiplicity one, and so Q_p^s has $(p-1)/s$ irreducible factors, each appearing exactly s times. $\square$

Example 7.3. If $p|(q-1)$, then $s = 1$, and $\sigma : \mathbb{F}_q \to \mathbb{F}_q$ is given by $\sigma(1) = r$, where $r^p = 1 \pmod{q}$ and $r \neq 1$. Thus, $M_{p,q} = \langle a, b \mid a^p = b^q = 1, a^{-1}ba = b^r \rangle$ is the metacyclic group of order pq, and $\mathrm{Aut}(M_{p,q}) \cong M_{q-1,q}$. Well-known examples are the dihedral groups $D_{2q} = M_{2,q}$, and, in particular, the symmetric group $S_3 = D_6$.

If $p = 3$ and $q = 2$, then $s = 2$ and $\sigma = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. The group $M_{3,4} = \mathbb{Z}_2^2 \rtimes_\sigma \mathbb{Z}_3$ is isomorphic to the alternating group A_4, and $\mathrm{Aut}(M_{3,4}) \cong S_4$.

7.4. **Metabelian representations.** We now study the homomorphisms from a finitely-presented group G to the metabelian group $\Gamma = M_{p,q^s}$. Our approach is modelled on that of Fox [14], where similar results are obtained in the case where G is a link group (with the Wirtinger presentation), and $\Gamma = M_{p,q}$ is metacyclic.

Let $\phi : F_\ell \twoheadrightarrow G$ be a presenting homomorphism, with $F_\ell = \langle x_1, \dots, x_\ell \rangle$. Let $\Gamma = B \rtimes_\sigma C$ be a semidirect product of abelian groups, with monodromy homomorphism $\sigma : C \to \mathrm{Aut}(B)$. Denote also by σ the linear extension to group-rings, $\sigma : \mathbb{Z}C \to \mathrm{End}(B)$. Finally, let $\rho : G \to C$ be a homomorphism, and set $\bar\rho = \rho \circ \phi : F_\ell \to C$.

Lemma 7.5. *Suppose $\lambda : F_\ell \to \Gamma$ is a lift of $\bar\rho$, given on generators by $\lambda(x_i) = b_i\bar\phi(x_i)$. If $\lambda(w) = b\bar\rho(w)$, then the following equality holds in B:*

$$b = \sum_{j=1}^{\ell} \sigma\bar\rho\left(\tfrac{\partial w}{\partial x_j}\right)(b_j).$$

Proof. The proof is by induction on the length of the word $w \in F_\ell$. If $w = 1$, the equality holds trivially. Suppose $w = ux_i^e$, where $e = \pm 1$. Put $\lambda(u) = b'\bar\rho(u)$. We then have: $b\bar\rho(w) = \lambda(w) = \lambda(ux_i^e) = b'\bar\rho(u)(b_i\bar\rho(x_i))^e$. Rewriting this last word in normal form (in the semidirect product $\Gamma = B \rtimes_\sigma C$), we obtain the following equality (in the additive group B):

$$b = b' + e\,\sigma\bar\rho\big(ux_i^{(e-1)/2}\big)(b_i). \tag{7.2}$$

Taking Fox derivatives of $w = ux_i^e$, and applying $\bar\rho$, gives:

$$\bar\rho\big(\tfrac{\partial w}{\partial x_j}\big) = \bar\rho\big(\tfrac{\partial u}{\partial x_j}\big) + e\,\bar\rho\big(ux_i^{(e-1)/2}\big)\delta_{ij}. \tag{7.3}$$

Now apply σ, evaluate at b_j, and sum over j:

$$\begin{aligned}\sum_{j=1}^{\ell} \sigma\big(\bar\rho\big(\tfrac{\partial w}{\partial x_j}\big)\big)(b_j) &= \sum_{j=1}^{\ell} \sigma\big(\bar\rho\big(\tfrac{\partial u}{\partial x_j}\big)\big)(b_j) + e\,\sigma\big(\bar\rho\big(ux_i^{(e-1)/2}\big)\big)(b_i)\\ &= b' + e\,\sigma\big(\bar\rho\big(ux_i^{(e-1)/2}\big)\big)(b_i) \quad \text{by induction hypothesis}\\ &= b \quad \text{by (7.2).}\end{aligned}$$

□

For p and q distinct primes, with $s = \operatorname{ord}_p(q)$, let $M_{p,q^s} = \mathbb{Z}_q^s \rtimes_\sigma \mathbb{Z}_p$ be the split metabelian group defined in 7.1. Let b be a generator of the cyclic group $\mathbb{Z}_p$. Viewing $\mathbb{Z}_q^s$ as the additive group of the field $\mathbb{K} = \mathbb{F}_q(\xi)$, where $\xi \in \mathbb{K}^*$ is a primitive p^{th} root of unity, we may take $\sigma(b) = \cdot\xi \in \operatorname{Aut}(\mathbb{K})$. In particular, this identifies $\mathbb{Z}_p$ as a subgroup of $\mathbb{K}^*$, and thus, $\operatorname{Hom}(G, \mathbb{Z}_p)$ as a subset of the character variety $\operatorname{Hom}(G, \mathbb{K}^*)$.

Proposition 7.6. *The number of homomorphisms, respectively epimorphisms from the finitely-presented group G to the metabelian group M_{p,q^s} is given by*

$$|\operatorname{Hom}(G, M_{p,q^s})| = \sum_{\rho \in \operatorname{Hom}(G,\mathbb{Z}_p)} q^{s d_{\mathbb{K}}(\rho) + s},$$
$$|\operatorname{Epi}(G, M_{p,q^s})| = \sum_{\mathbf{1} \neq \rho \in \operatorname{Hom}(G,\mathbb{Z}_p)} q^{s}(q^{s d_{\mathbb{K}}(\rho)} - 1).$$

where $\mathbb{K} = \mathbb{F}_{q^s}$, and the sums are over representations $\rho : G \to \mathbb{Z}_p \subset \mathbb{K}^*$.

Proof. Let $G = \langle x_1, \dots, x_\ell \mid r_1, \dots, r_m \rangle$ be a presentation for G. Let $\rho : G \to \mathbb{Z}_p$ be a representation, given by $\rho(x_i) = b^{\beta_i}$. We want to lift it to a representation $\lambda : G \to M_{p,q^s}$. Such a representation is given by $\lambda(x_i) = u_i b^{\beta_i}$, where $u_i \in \mathbb{Z}_q^s$. In view of Lemma 7.5, we must solve the following system of equations over $\mathbb{K} = \mathbb{F}_q(\xi)$:

$$\sum_{j=1}^{\ell} A_{k,j}(\xi^{\beta_1}, \dots, \xi^{\beta_n}) \cdot u_j = 0, \quad 1 \le k \le m, \tag{7.4}$$

where $A_G = (A_{k,j})$ is the Alexander matrix of G. This system has $q^{sd_{\mathbb{K}}(\rho)+s}$ solutions. Starting now with a non-trivial representation $\rho : G \to \mathbb{Z}_p$, all such solutions give rise to surjective representations $\lambda : G \to M_{p,q^s}$, except q^s of them, which give rise to abelian representations. $\square$

This proposition, together with Lemma 7.2(3), imply the following.

Theorem 7.7. *If p and q are distinct primes, and $s = \operatorname{ord}_p(q)$, then:*

$$\delta_{M_{p,q^s}}(G) = \frac{p-1}{s(q^s-1)} \sum_{d\geq 1} \beta^{(q)}_{p,d}(G)(q^{sd}-1).$$

Example 7.8. For free groups, Theorem 7.7 gives:

$$\delta_{M_{p,q^s}}(F_n) = \frac{(p^n-1)(q^{s(n-1)}-1)}{s(q^s-1)}, \tag{7.5}$$

since $\beta^{(q)}_{p,n-1}(F_n) = \frac{p^n-1}{p-1}$, and the other terms in the sum vanish. In particular, this recovers formulas (2.9) (when $p \mid q-1$) and (2.10) (when $p=3$, $q=2$). For a product of free groups, we get:

$$\delta_{M_{p,q^s}}(F_{n_1} \times \cdots \times F_{n_k}) = \frac{\sum_{i=1}^{k}(p^{n_i}-1)(q^{s(n_i-1))}-1)}{s(q^s-1)}. \tag{7.6}$$

Example 7.9. For orientable surface groups of genus $g \geq 1$, Theorem 7.7 gives:

$$\delta_{M_{p,q^s}}(G) = \frac{(p^{2g}-1)(q^{2s(g-1)}-1)}{s(q^s-1)}. \tag{7.7}$$

For non-orientable surface groups of genus $n \geq 1$, we get:

$$\delta_{D_{2q}}(G) = \frac{(2^n-2)(q^{n-2}-1)+q^{n-1}-1}{q-1}, \tag{7.8}$$

$$\delta_{M_{p,q^s}}(G) = \frac{(p^{n-1}-1)(q^{s(n-2)}-1)}{s(q^s-1)}. \tag{7.9}$$

Table 1 gives the values of $\delta_\Gamma(G)$, for some of the groups G in Examples 7.8 and 7.9, and for some finite groups Γ of small order.

8. Counting finite-index subgroups

We now discuss some other invariants of a finitely-generated group G, obtained by counting finite-index subgroups of G in various ways. If G is finitely-presented, and the index is low, these invariants can be computed from the characteristic varieties of G, and some simple homological data.

$G \setminus \Gamma$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	$\mathbb{Z}_8$	S_3	A_4	$M_{3,7}$
F_2	3	4	1	6	3	12	3	4	8
F_3	7	13	7	28	42	112	28	65	208
F_4	15	40	35	120	420	960	195	840	$4,560$
$F_2 \times F_1$	7	13	7	28	42	112	3	4	8
$F_2 \times F_2$	15	40	35	120	420	960	6	8	16
$F_3 \times F_1$	15	40	35	120	420	960	28	65	208
$F_3 \times F_2$	31	121	155	496	$3,720$	$7,936$	31	69	216
$\pi_1(\#_2 T^2)$	15	40	35	120	420	960	60	200	640
$\pi_1(\#_3 T^2)$	63	364	651	$2,016$	$31,248$	$64,512$	$2,520$	$30,940$	$291,200$
$\pi_1(\#_4 T^2)$	255	$3,280$	$10,795$	$32,640$	$2,072,640$	$4,177,920$	$92,820$	$4,477,200$	$128,628,480$
$\pi_1(\#_2 \mathbb{RP}^2)$	3	1	1	2	1	2	1	0	0
$\pi_1(\#_3 \mathbb{RP}^2)$	7	4	7	12	18	24	10	4	8
$\pi_1(\#_4 \mathbb{RP}^2)$	15	13	35	56	196	224	69	65	208
$\pi_1(\#_5 \mathbb{RP}^2)$	31	40	155	240	$1,800$	$1,920$	430	840	$4,560$

TABLE 1. Γ-Hall invariants of some finitely presented groups G.

8.1. **Subgroups of finite index.** For each positive integer k, let

$$a_k(G) = \text{number of index } k \text{ subgroups of } G. \tag{8.1}$$

Also, let $h_l(G) = \sigma_{S_l}(G)$ be the number of homomorphisms from G to the symmetric group S_l. The following well-known formula of Marshall Hall [17] (see also [29]) computes a_k in terms of $h_1, \dots, h_k$ (starting from $a_1 = h_1 = 1$):

$$a_k(G) = \frac{1}{(k-1)!}\, h_k(G) - \sum_{l=1}^{k-1} \frac{1}{(k-l)!}\, h_{k-l}(G) a_l(G). \tag{8.2}$$

For the free group $G = F_n$, we have $h_k(F_n) = (k!)^n$, and so, as noted by M. Hall,

$$a_k(F_n) = k(k!)^{n-1} - \sum_{l=1}^{k-1} ((k-l)!)^{n-1} a_l(F_n). \tag{8.3}$$

For the free abelian group $G = \mathbb{Z}^n$, a result of Bushnell and Reiner [6] gives $a_k(\mathbb{Z}^n)$ recursively, starting from $a_k(\mathbb{Z}) = 1$:

$$a_k(\mathbb{Z}^n) = \sum_{d \mid k} a_d(\mathbb{Z}^{n-1}) \left(\tfrac{k}{d}\right)^{n-1} \tag{8.4}$$

(see [27] for a simple proof, using the Hermite normal form of integral matrices). Equivalently, $\zeta_{\mathbb{Z}^n}(s) = \prod_{i=0}^{n-1} \zeta(s-i)$, where $\zeta_G(s) = \sum_{k=1}^{\infty} a_k(G) k^{-s}$ is the zeta

function of the group G, and $\zeta(s)$ is the classical Riemann zeta function (see [29] for a detailed discussion).

For surface groups G, the numbers $a_k(G)$ were computed by Mednykh [35].

As an application of our methods, we express the number of index 2 and 3 subgroups of a finitely-presented group, in terms of its characteristic varieties.

Theorem 8.2. *Let G be a finitely-presented group. Set $n_p = b_1^{(p)}(G)$. Then, the number of index 2 and 3 subgroups of G is given by:*

$$a_2(G) = 2^{n_2} - 1,$$
$$a_3(G) = \tfrac{1}{2}(3^{n_3} - 1) + \tfrac{3}{2}\sum_{d\geq 1}\beta_{2,d}^{(3)}(G)(3^d - 1).$$

Proof. Clearly, $a_2 = h_2 - 1 = \delta_{\mathbb{Z}_2}$, and the first identity follows from Theorem 3.1.

M. Hall's formula (8.2) gives $a_3 = \frac{1}{2}h_3 - \frac{3}{2}h_2 + 1$. Recall that the subgroup lattice of the symmetric group $S_3 = M_{2,3}$ is $L(S_3) = \{1, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, S_3\}$ (see Figure 1). Thus, P. Hall's formula (2.5) gives $h_3 = 1 + 3\delta_{\mathbb{Z}_2} + 2\delta_{\mathbb{Z}_3} + 6\delta_{S_3}$. Using Theorem 3.1, we get:

$$a_3(G) = \tfrac{1}{2}(3^{n_3} - 1) + 3\delta_{S_3}(G). \tag{8.5}$$

The second identity now follows from Theorem 7.7. □

For example, $a_3(F_n) = 3(3^{n-1} - 1)2^{n-1} + 1$, which agrees with M. Hall's computation. Also, $a_3(\pi_1(\#_g T^2)) = 3(3^{2g-2} - 1)(2^{2g-1} + 1) + 4$ and $a_3(\pi_1(\#_n \mathbb{RP}^2)) = 3(3^{n-2} - 1)(2^{n-1} + 1) + 4$, which agrees with Mednykh's computation.

Remark 8.3. To compute $a_4(G)$ by the same method, one needs to do more work. Indeed, $a_4 = \frac{1}{6}h_4 - \frac{2}{3}h_3 - \frac{1}{2}h_2^2 + 2h_2 - 1$, and

$$h_4 = 1 + 9\delta_{\mathbb{Z}_2} + 8\delta_{\mathbb{Z}_3} + 6\delta_{\mathbb{Z}_4} + 24\delta_{\mathbb{Z}_2^2} + 24\delta_{S_3} + 24\delta_{D_8} + 24\delta_{A_4} + 24\delta_{S_4}. \tag{8.6}$$

All the terms in the sum can be computed as above, except those corresponding to $D_8 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$, and $S_4 = \mathbb{Z}_2^2 \rtimes \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, for which other techniques are needed.

8.4. **Normal subgroups of finite index.** For each positive integer k, let $a_k^{\triangleleft}(G)$ be the number of index k, normal subgroups of G. We then have:

$$\alpha_k^{\triangleleft}(G) = \sum_{|\Gamma|=k} \delta_\Gamma(G). \tag{8.7}$$

Using this formula, and our previous formulas for the Hall invariants, we can compute $\alpha_k^{\triangleleft}(G)$ in terms of homological data, provided k has at most two factors.

Theorem 8.5. *Let G be a finitely-presented group.*

(1) *If p is prime, then*

$$a^{\triangleleft}_{p}(G) = \tfrac{p^n-1}{p-1}$$
$$a^{\triangleleft}_{p^2}(G) = \tfrac{(p^n-1)(p^{n-1}-1)}{(p^2-1)(p-1)} + \tfrac{p^{n-1}(p^m-1)}{p-1}$$

where $n = b_1^{(p)}(G)$ *and* $m = \dim_{\mathbb{Z}_p}(p \cdot H_1(G, \mathbb{Z}_{p^2})) \otimes \mathbb{Z}_p$.

(2) *If p and q are distinct primes, then*

$$a^{\triangleleft}_{pq}(G) = \begin{cases} \tfrac{(p^n-1)(q^m-1)}{(p-1)(q-1)} & \text{if} \quad p \nmid q-1 \\ \tfrac{(p^n-1)(q^m-1)}{(p-1)(q-1)} + \tfrac{p-1}{q-1}\sum_{d\geq 1} \beta^{(q)}_{p,d}(G)(q^d-1) & \text{if} \quad p \mid q-1 \end{cases}$$

where $n = b_1^{(p)}(G)$ *and* $m = b_1^{(q)}(G)$.

Proof. The only group of order p is $\mathbb{Z}_p$; the only groups of order p^2 are $\mathbb{Z}_p^2$ and $\mathbb{Z}_{p^2}$; the only groups of order pq are $\mathbb{Z}_{pq}$ (if $p \nmid q-1$), and $\mathbb{Z}_{pq}$ and $M_{p,q}$ (if $p \mid q-1$). The formulas follow from (8.7) and Theorems 3.1 and 7.7. □

Note that these formulas compute $a^{\triangleleft}_k$ for all $k \leq 15$, except for $k = 8$ and $k = 12$. To compute $a^{\triangleleft}_8$, one would need to know δ_{D_8} and δ_{Q_8}, where Q_8 is the quaternion group; for $a^{\triangleleft}_{12}$, one would need $\delta_{D_{12}}$ and $\delta_{D'_{12}}$, where $D'_{12} = \mathbb{Z}_3 \rtimes \mathbb{Z}_4$ is the dicylic group of order 12.

Remark 8.6. We may also define $\alpha_k(G)$ to be the number of index k, normal subgroups $K \triangleleft G$, with G/K abelian. That is,

$$\alpha_k(G) = \sum_{\substack{\Gamma \text{ abelian} \\ |\Gamma|=k}} \delta_\Gamma(G). \tag{8.8}$$

Clearly, $\alpha_k(G) = a_k(H_1(G))$. In particular, if $H_1(G) = \mathbb{Z}^n$, then $\alpha_k(G) = a_k(\mathbb{Z}^n)$ is given by the recursion (8.4).

Finally, let $c_k(G)$ be the number of conjugacy classes of index k subgroups of G. If p is a prime, then clearly $a_p(G) = pc_p(G) - (p-1)a^{\triangleleft}_p(G)$. Hence, if $n = b_1^{(p)}(G)$, we have:

$$c_p(G) = \tfrac{p^n + a_p(G) - 1}{p}. \tag{8.9}$$

Remark 8.7. The following formula of Stanley [40, (5.125)] holds: $a_k(G \times \mathbb{Z}) = \sum_{d|k} dc_k(G)$. Hence, if p is a prime, and $n = b_1^{(p)}(G)$, we have:

$$a_p(G \times \mathbb{Z}) = a_p(G) + p^n. \tag{8.10}$$

9. Arrangements of complex hyperplanes

A *(complex) hyperplane arrangement* is a finite collection of codimension 1 affine subspaces in a complex vector space. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement of n hyperplanes in $\mathbb{C}^\ell$. A defining polynomial for $\mathcal{A}$ may be written as $f = f_1 \cdots f_n$, where f_i are (distinct) linear forms. Choose coordinates $(z_1, \dots, z_\ell)$ in $\mathbb{C}^\ell$ so that $H_n = \ker(z_\ell)$. The *decone* $\mathcal{A}^* = \mathbf{d}\mathcal{A}$ (corresponding to this choice) is the affine arrangement in $\mathbb{C}^{\ell-1}$ with defining polynomial $f^* = f(z_1, \dots, z_{\ell-1}, 1)$. If $X(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$ is the complement of $\mathcal{A}$, then $X(\mathcal{A}) \cong X(\mathcal{A}^*) \times \mathbb{C}^*$.

Let $G = G(\mathcal{A}) = \pi_1(X(\mathcal{A}))$ be the fundamental group of the complement of $\mathcal{A}$. Then $G(\mathcal{A}) \cong G(\mathcal{A}^*) \times \mathbb{Z}$. Let m be the number of multiple points in a generic 2-section of $\mathcal{A}^*$. The group $G^* = G(\mathcal{A}^*)$ admits a finite presentation of the form

$$G^* = \langle x_1, \dots, x_{n-1} \mid \alpha_j(x_i) = x_i, \text{ for } j = 1, \dots, m \text{ and } i = 1, \dots, n-1\rangle,$$

where $\alpha_1, \dots, \alpha_m$ are the "braid monodromy" generators—pure braids on $n-1$ strings, acting on $F_{n-1} = \langle x_1, \dots, x_{n-1}\rangle$ via the Artin representation, see [8] for details and further references. In particular, $H_1(G) = \mathbb{Z}^n$.

Let $V_d(G, \mathbb{K})$ be the characteristic varieties of the arrangement $\mathcal{A}$ (over the field $\mathbb{K}$). If $\mathbb{K} = \mathbb{C}$, the following facts are known:

(a) The components of $V_d(G, \mathbb{C})$ are subtori of the character torus $(\mathbb{C}^*)^n$, possibly translated by roots of unity (cf. [1]).
(b) The tangent cone at $\mathbf{1}$ to $V_d(G, \mathbb{C})$ coincides with the resonance variety $R_d(G, \mathbb{C})$; thus, the components of $R_d(G, \mathbb{C})$ are linear subspaces of $\mathbb{C}^n$ (cf. [9, 24]).

We do not know whether (a) holds if $\mathbb{C}$ is replaced by a field $\mathbb{K}$ of positive characteristic. On the other hand, the first half of (b) can easily fail in that case. We refer to [9, 12, 24, 26] for methods of computing the (complex) characteristic and resonance varieties of hyperplane arrangements, and to [41] for further details on the examples below.

Example 9.1. Let $\mathcal{A}$ be the braid arrangement in $\mathbb{C}^3$, with defining polynomial $f = xyz(x-y)(x-z)(y-z)$. The fundamental group is $G = P_4$, the pure braid group on 4 strands.

For any field $\mathbb{K}$, the variety $V_1(G, \mathbb{K}) \subset (\mathbb{K}^*)^6$ has five components, all 2-dimensional: four 'local' components, corresponding to triple points, and one 'non-local' component, corresponding to an (essential) neighborly partition of the matroid. The components meet only at the origin, $\mathbf{1} = (1, \dots, 1)$. Moreover, $V_2(G, \mathbb{K}) = \{\mathbf{1}\}$.

Let p be a prime, and $q = 0$, or a prime distinct from p. From Theorem 6.5, we get:

$$\beta^{(q)}_{p,0}(G) = (p+1)(p^4 + p^2 - 4), \quad \beta^{(q)}_{p,1}(G) = 5(p+1),$$

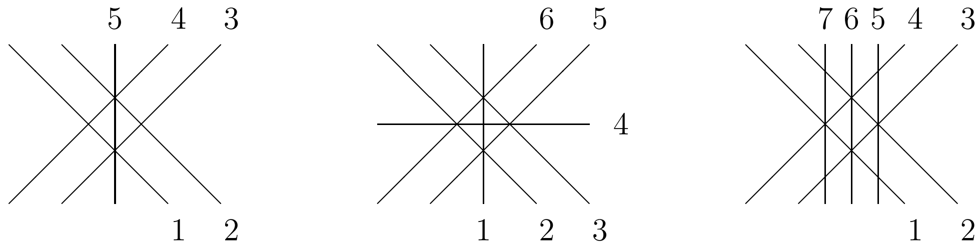

FIGURE 2. Decones of braid, non-Fano, and deleted B_3 arrangements

and $\beta^{(q)}_{p,d}(G) = 0$, for $d > 1$. Thus, $\beta^{(q)}_p(G) = (5(p+1))$, for all q. Theorem 7.7 now gives:

$$\delta_{M_{p,q^s}}(G) = 5(p^2-1)/s.$$

Example 9.2. Let $\mathcal{A}$ be the realization of the non-Fano plane, with defining polynomial $f = xyz(x-y)(x-z)(y-z)(x+y-z)$. The group $G = G^* \times \mathbb{Z}$ is given by the braid monodromy generators $\{A_{345},\ A_{125}^{A_{35}A_{45}},\ A_{14}^{A_{34}},\ A_{136},\ A_{246}^{A_{34}A_{36}}\}$, where A_I denotes the full twist on the strands indexed by I, and $x^y = y^{-1}xy$.

Let $\mathbb{K}$ be a field. The variety $V_1(G,\mathbb{K}) \subset (\mathbb{K}^*)^7$ has nine 2-dimensional components: 6 corresponding to triple points, and 3 corresponding to braid sub-arrangements. All components intersect at the origin. If $\operatorname{char}\mathbb{K} \neq 2$, the 3 non-local components also intersect at a point of order 2, belonging to V_2. Thus, $\beta^{(q)}_2 = (24,1)$, and $\beta^{(q)}_p = (9(p+1))$. Hence:

$$\delta_{D_{2q}} = q + 25 \quad \text{and} \quad \delta_{M_{p,q^s}} = 9(p^2-1)/s, \text{ for } p > 2.$$

Example 9.3. The arrangement defined by $f = xyz(x-y)(x-z)(y-z)(x-y-z)(x-y+z)$ is a deletion of the reflection arrangement of type B_3. The group G is isomorphic to $(F_4 \rtimes_\alpha F_3) \times \mathbb{Z}$, where $\alpha : F_3 \to P_4$ is given by $\alpha = \{A_{23},\ A_{13}^{A_{23}}A_{24},\ A_{14}^{A_{24}}\}$.

The variety $V_1(G,\mathbb{K}) \subset (\mathbb{K}^*)^8$ has eleven 2-dimensional components (6 corresponding to triple points, and 5 to braid sub-arrangements), and one 1-dimensional component,

$$\begin{aligned} C &= \{t_4 - t_1 = t_3 - t_2 = t_5 - t_1^2 = t_7 - t_2^2 = t_6 + 1 = t_8 + 1 = t_1t_2 + 1 = 0\} \\ &= \{(t, -t^{-1}, -t^{-1}, t, t^2, -1, t^{-2}, -1) \mid t \in \mathbb{K}^*\}. \end{aligned}$$

The variety $V_2(G,\mathbb{K})$ has a 3-dimensional component (corresponding to a quadruple point). Let $q = \operatorname{char}\mathbb{K}$. There are two cases to consider:

If $q \neq 2$, then $V_2(G,\mathbb{K})$ also has two isolated points of order 2. Moreover, C does not pass through the origin, though it meets the other non-local components at the

two isolated points of V_2. It follows that $\beta_2^{(q)} = (27, 9)$ and $\beta_p^{(q)} = (11(p+1), p^2 + p + 1)$, for p odd.

If $q = 2$, then all the components of $V_d(G, \mathbb{K})$ pass through the origin. Note that the Galois field $\mathbb{K} = \mathbb{F}_4$, obtained by adjoining to $\mathbb{F}_2$ the primitive 3^{rd} root of unity $\omega = e^{2\pi\,\mathrm{i}/3}$, is sufficiently large with respect to $\mathbb{Z}_3$. The representation $\mu : G \to \mathbb{Z}_3 = \mathbb{F}_4^*$ given by $\mu = (\omega, \omega^2, \omega^2, \omega, \omega^2, 1, \omega, 1)$ belongs to $C \subset V_1(G, \mathbb{F}_4)$, but does not belong to $V_1(G, \mathbb{C})$. Thus, by Corollary 6.3, $b_1(K_\mu) = 8$ and $b_1^{(2)}(K_\mu) = 10$. Moreover, $b_1^{(2)}(K_\lambda) = b_1(K_\lambda)$, unless $\lambda = \mu$ or $\bar{\mu}$. It follows that $\beta_{p,d}^{(2)} = \beta_{p,d}$, except for $\beta_{3,1}^{(2)} = \beta_{3,1} + 1 = 45$.

Using now Theorem 7.7, we conclude:

$$\begin{aligned} \delta_{D_{2q}} &= 9(q+4), \quad \delta_{A_4} = 110, \\ \delta_{M_{p,q^s}} &= ((p^3 - 1)q^s + p^3 + 11p^2 - 12)/s, \text{ otherwise.} \end{aligned}$$

Remark 9.4. Using the formulas from Section 8, we may compute the number of low-index subgroups of arrangement groups. For example, if $|\mathcal{A}| = n$, then $a_3(G) = \frac{1}{2}(3^n - 1) + 3\delta_{S_3}(G)$. Thus, the braid arrangement has $a_3 = 409$, the non-Fano plane has $a_3 = 1,177$, and the deleted B_3 arrangement has $a_3 = 3,469$.

The idea to use the count of index 3 subgroups as an invariant for hyperplane arrangement groups originates with the (unpublished) work of M. Falk and B. Sturmfels. These authors considered a pair of non-lattice-isomorphic arrangements of 9 planes in $\mathbb{C}^3$. The respective groups are in fact isomorphic (see [8, Example 7.5]). In each case, the variety V_1 has twelve 2-dimensional components (8 corresponding to triple points, and 4 to braid sub-arrangements), and V_2 has one 3-dimensional component (corresponding to a quadruple point). Hence: $\delta_{S_3} = \frac{1}{2}(12(2^2 - 1)(3 - 1) + (2^3 - 1)(3^2 - 1)) = 64$, $a_3 = 10,033$, and $c_3 = 9,905$.

10. Arrangements of transverse planes in $\mathbb{R}^4$

A 2-*arrangement* in $\mathbb{R}^4$ is a finite collection $\mathcal{A} = \{H_1, \dots, H_n\}$ of transverse planes through the origin of $\mathbb{R}^4$. In coordinates (z, w) for $\mathbb{R}^4 = \mathbb{C}^2$, a defining polynomial for $\mathcal{A}$ may be written as $f = f_1 \cdots f_n$, where $f_i(z, w) = a_i z + b_i \bar{z} + c_i w + d_i \bar{w}$. A generic section of $\mathcal{A}$ by an affine 3-plane in $\mathbb{R}^4$ yields a configuration of n skew lines in $\mathbb{R}^3$; conversely, coning such a configuration yields a 2-arrangement in $\mathbb{R}^4$.

The complement of the arrangement, $X(\mathcal{A}) = \mathbb{R}^4 \setminus \bigcup_{i=1}^n H_i$, deform-retracts onto the complement of the link $L(\mathcal{A}) = \mathbb{S}^3 \cap \bigcup_{i=1}^n H_i$. The link $L(\mathcal{A})$ is the closure of a pure braid $\beta \in P_n$. The fundamental group of the complement, $G(\mathcal{A}) = \pi_1(X(\mathcal{A}))$, has the structure of a semidirect product of free groups: $G(\mathcal{A}) = F_{n-1} \rtimes_{\xi^2} \mathbb{Z}$, where ξ is a certain pure braid in P_{n-1}, determined by β, see [32]. Moreover, $X(\mathcal{A})$ is an Eilenberg-MacLane space $K(G, 1)$.

A 2-arrangement $\mathcal{A}$ is called *horizontal* if it admits a defining polynomial of the form $f = \prod_{i=1}^{n}(z+a_i w+b_i\bar{w})$, with a_i, b_i real. From the coefficients of f, one reads off a permutation $\tau \in S_n$. Conversely, given $\tau \in S_n$, choose real numbers $a_1 < \cdots < a_n$ and $b_{\tau_1} < \cdots < b_{\tau_n}$. Then the polynomial $f = \prod_{i=1}^{n}(z - \frac{a_i+b_i}{2}w - \frac{a_i-b_i}{2}\bar{w})$ defines a horizontal arrangement, $\mathcal{A}(\tau)$, whose associated permutation is τ. The braid ξ corresponding to $\mathcal{A} = \mathcal{A}(\tau)$ can be combed as $\xi = \xi_2 \cdots \xi_{n-1}$, where $\xi_j = \prod_{i=1}^{j-1} A_{i,j}^{e_{i,j}}$ and $e_{i,j} = 1$ if $\tau_i > \tau_j$, and $e_{i,j} = 0$, otherwise.

For $n \leq 5$, all 2-arrangements are horizontal. For $n = 6$, there are 4 non-horizontal arrangements: $\mathcal{L}$, $\mathcal{M}$, and their mirror images. These arrangements were introduced by Mazurovskiĭ in [34]; further details about them can be found in [32]. For $n = 7$, there are 13 non-horizontal arrangements, see [2].

The (complex) characteristic varieties of 2-arrangement groups $G = G(\mathcal{A})$ were studied in [32]. Note that $V_1(G, \mathbb{C})$ is the hypersurface in $(\mathbb{C}^*)^n$ defined by the Alexander polynomial of the link $L(\mathcal{A})$; see Penne [36] for another way to compute this polynomial. We refer to [33] for more information on the resonance varieties of 2-arrangements.

10.1. **Computations of Hall invariants.** We now show how to compute the distribution $\beta_p^{(q)}(G)$ of mod q Betti numbers of index p normal subgroups, and the Hall invariants $\delta_{M_{p,q^s}}(G)$, for some 2-arrangement groups G.

Example 10.2. Let $\mathcal{A} = \mathcal{A}(2134)$ be the horizontal arrangement defined by the polynomial $f = zw(z-w)(z-2\bar{w})$. The group of the complement is $G = F_3 \rtimes_{\xi^2} \mathbb{Z}$, where $\xi = A_{1,2}$. The characteristic varieties $V_d = V_d(G, \mathbb{K})$ are given by:

$$\begin{aligned}
V_1 &= \{t_4 = 1\} \cup \{t_4 = t_2^2\},\\
V_2 &= \{t_4 = 1, t_2 = -1\} \cup \{t_4 = t_2 = t_1 = 1\} \cup \{t_4 = t_2 = t_3 = 1\},\\
V_3 &= \{(1,1,1,1)\}.
\end{aligned}$$

Counting 2-torsion points on $V_d(G, \mathbb{K})$, we see that $\operatorname{Tors}_{2,1}(G,\mathbb{K}) \setminus \operatorname{Tors}_{2,2}(G,\mathbb{K}) = \{(-1,1,-1,1)\}$, and $\operatorname{Tors}_{2,2}(G,\mathbb{K}) \setminus \operatorname{Tors}_{2,3}(G,\mathbb{K}) = \{(1,-1,\pm1,1), (-1,-1,\pm1,1), (1,1,-1,1), (-1,1,1,1)\}$, provided $\operatorname{char}\mathbb{K} \neq 2$. Hence, $\beta_2^{(q)} = (1,6)$, if $q \neq 2$. For an odd prime p, the count of p-torsion points yields $\beta_p^{(q)} = (2p^2+p-1, 2)$, if $q \nmid 2p$.

Now suppose $q = 2$. A sufficiently large field for the group $\mathbb{Z}_p$ is $\mathbb{F}_{2^s} = \mathbb{F}_2(\zeta)$, where $s = \operatorname{ord}_p(2)$ and $\zeta = e^{2\pi\,\mathrm{i}/p}$. We have $V_2(G, \mathbb{F}_{2^s}) = \{t_4 = t_2 = 1\}$, and so $\operatorname{Tors}_{p,2}(G, \mathbb{F}_{2^s})$ also contains the points $(\zeta^j, 1, \zeta, 1)$, for $0 < j < p$. Hence, $\beta_p^{(2)} = (2p^2, p+1)$. By Theorem 7.7:

$$\begin{aligned}
\delta_{D_{2q}} &= 6q + 7, &&\\
\delta_{M_{p,2^s}} &= (p-1)(2p^2 + (p+1)(2^s+1))/s, && \text{for } p > 2,\\
\delta_{M_{p,q^s}} &= (p-1)(2p^2 + p + 2q^s + 1)/s, && \text{for } p, q > 2.
\end{aligned}$$

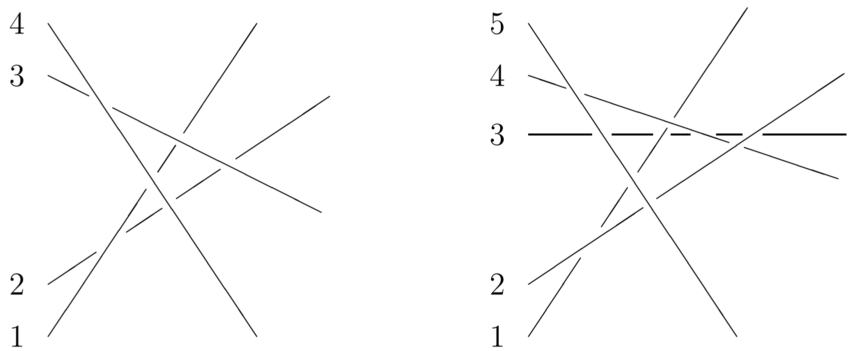

FIGURE 3. Generic 3-sections of $\mathcal{A}(2134)$ and $\mathcal{A}(31425)$

Example 10.3. Let $\mathcal{A} = \mathcal{A}(31425)$. A defining polynomial is $f = z(z-w)(z-2w)(z+\frac{3}{2}w-\frac{5}{2}\bar{w})(z-\frac{1}{2}w-\frac{5}{2}\bar{w})$. The group is $G = F_4 \rtimes_{\xi^2} \mathbb{Z}$, where $\xi = A_{1,3}A_{2,3}A_{2,4}$. If $\operatorname{char}\mathbb{K} \neq 2$, the varieties $V_d = V_d(G,\mathbb{K})$ are as follows:

$$
\begin{aligned}
V_1 = \{&t_2^2t_3^2t_4^2 - t_1^2t_2^2t_3^2t_5 - t_2^2t_4^2t_5 - t_1^2t_2^2t_4^2t_5 - t_1^2t_3^2t_4^2t_5 + {t_2}^4t_5^2 + t_1^2t_3^2t_5^2 + t_2^2t_4^2t_5^2 - t_1^2t_2^2t_5^3 - {t_2}^4t_4^2t_5 + \\
&t_2^2t_3^2t_5^2 + t_1^2t_2^2t_3^2t_5^2 + 2t_5(t_1^2t_2^2t_3t_4 + t_2^3t_3t_4 - t_1t_2^3t_3t_4 - t_1t_2^2t_3t_4 + t_1t_2t_3^2t_4 - t_1^2t_2t_3^2t_4 - t_2^2t_3^2t_4 + \\
&t_1t_2^2t_3^2t_4 + t_1t_2^2t_4^2 - t_2^3t_4^2 + t_1t_2^3t_4^2 - t_1t_2t_3t_4^2 + t_1^2t_2t_3t_4^2 + t_2^2t_3t_4^2 - t_1t_2^2t_3t_4^2 + t_1t_2^2t_3t_5 - \\
&t_1t_2t_3^2t_5 - t_1^2t_2^2t_3t_5 - t_2^3t_3t_5 + t_1t_2^3t_3t_5 + t_1^2t_2t_3^2t_5 - t_1t_2^2t_3^2t_5 - t_1t_2^2t_4t_5 + t_1^2t_2^2t_4t_5 + t_2^3t_4t_5 - \\
&t_1t_2^3t_4t_5 + t_1t_2t_3t_4t_5 - t_1^2t_2t_3t_4t_5 - t_2^2t_3t_4t_5 + t_1t_2^2t_3t_4t_5) = 0\},
\end{aligned}
$$

$$
\begin{aligned}
V_2 = \{&t_1t_2^2t_3 + t_1^2t_3^2 - t_1t_2t_3^2 + t_1^2t_2t_3^2 + t_2^2t_4 - t_1t_2^2t_4 + t_2^3t_4 + t_1t_3t_4 - t_1^2t_3t_4 - t_2t_3t_4 + 4t_1t_2t_3t_4 - \\
&t_1^2t_2t_3t_4 - t_2^2t_3t_4 + t_1t_2^2t_3t_4 - t_1t_3^2t_4 + t_1^2t_3^2t_4 + t_1^2{t_2}^{-1}t_3^2t_4 + t_2t_4^2 - t_1t_2t_4^2 + t_2^2t_4^2 + t_1t_3t_4^2 = \\
&t_5 - t_4 + t_3 + {t_1}^{-1}t_4 + {t_1}^{-1}t_2t_4 + {t_2}^{-1}t_3t_4 = 0\} \cup \{(t,1,1,1,1)\} \cup \{(1,1,t,1,1)\} \cup \\
&\{(1,1,1,t,1)\} \cup \{(t,t,t,t,1)\} \cup \{(1,t,t,1,1)\} \cup \{(t,t,1,1,1)\} \cup \{(1,t,t^2,t^2,t^2)\} \cup \\
&\{(1,1,t,t,t^2)\} \cup \{(t,t,t,t^2,t^2)\},
\end{aligned}
$$

$$
\begin{aligned}
V_3 = \{&(-1,-1,-1,\pm1,1), (1,-1,-1,\pm1,1), (1,1,-1,\pm1,1), (-1,\pm1,1,1,1), (1,\pm1,1,1,1), \\
&(1,1,1,-1,1)\},
\end{aligned}
$$

$$
V_4 = \{(1,1,1,1,1)\}.
$$

We start by counting points of order 2 on these varieties. Inspection shows that $\operatorname{Tors}_{2,1}(G,\mathbb{K}) \setminus \operatorname{Tors}_{2,2}(G,\mathbb{K}) = \{(-1,1,-1,\pm1,1), (1,-1,1,-1,1), (-1,1,1,-1,1), (-1,-1,1,-1,1)\}$ and $\operatorname{Tors}_{2,2}(G,\mathbb{K}) = \operatorname{Tors}_{2,3}(G,\mathbb{K})$. The set $\operatorname{Tors}_{2,3}(G,\mathbb{K})$ consists of the 10 points in $V_3 \setminus \{\mathbf{1}\}$, except when $\mathbb{K} = \mathbb{Z}_3$, in which case it also contains $(-1,1,-1,1,-1)$. Hence, $\beta_2^{(q)} = (5,0,10)$, except for $\beta_2^{(3)} = (5,1,10)$. Therefore:

$$
\delta_{S_3} = 139 \quad \text{and} \quad \delta_{D_{2q}} = 5(2q^2+2q+3), \text{ for } q > 3.
$$

Similar computations hold for torsion points of order 3. We easily see that $\beta_3^{(q)} = (60,10)$, if $q \neq 2, 5,$ or 7. On the other hand, consider the field $\mathbb{K} = \mathbb{F}_7$, which is sufficiently large with respect to $\mathbb{Z}_3$ (if we identify the additive group of $\mathbb{F}_7$ with

the multiplicative subgroup of $\mathbb{C}^*$ generated by $\zeta = e^{2\pi\,\mathrm{i}/7}$, we may view $\mathbb{Z}_3$ as the subgroup $\langle\zeta^2\rangle \subset \mathbb{F}_7^*$). Now $\mathrm{Tors}_{3,1}(G, \mathbb{F}_7) \setminus (\mathrm{Tors}_{3,2}(G, \mathbb{F}_7) \cup \mathrm{Tors}_{3,1}(G, \mathbb{C}))$ consists of $(\zeta^2, \zeta^4, 1, \zeta^4, 1), (\zeta^2, \zeta^4, \zeta^2, \zeta^4, \zeta^2), (\zeta^2, 1, \zeta^2, \zeta^4, \zeta^2), (\zeta^2, 1, \zeta^4, 1, \zeta^2), (\zeta^2, 1, \zeta^4, \zeta^2, 1)$, together with their conjugates. Hence, $\beta_3^{(7)} = (65, 10)$. Similarly, $\beta_3^{(2)} = (41, 30)$ and $\beta_3^{(5)} = (70, 10)$. Therefore:

$$\delta_{A_4} = 191,\ \delta_{M_{3,5^2}} = 330,\ \delta_{M_{3,7}} = 290,\ \text{and } \delta_{M_{3,q^s}} = 20(q^s + 7)/s,\ \text{for } q > 7.$$

Remark 10.4. The resonance varieties $R_d(G, \mathbb{C})$ of the arrangement $\mathcal{A}(31425)$ were computed in [33, Example 6.5]. Comparing the answer given there with the one from Example 10.3, we see that the variety $R_2(G, \mathbb{C})$ has 10 irreducible components, whereas $V_2(G, \mathbb{C})$ has only 9 components passing through the origin (the tenth component, which does *not* contain $\mathbf{1}$, is *not* a translated torus). Thus, the tangent cone at $\mathbf{1}$ to $V_2(G, \mathbb{C})$ is *strictly* contained in $R_2(G, \mathbb{C})$. Another example where such a strict inclusion occurs (with G the group of a certain 4-component link) was given in [31, §2.3].

10.5. **Classification of 2-arrangement groups.** The rigid isotopy classification of configurations of $n \leq 7$ skew lines in $\mathbb{R}^3$ (and, thereby, of 2-arrangements of $n \leq 7$ planes in $\mathbb{R}^4$) was established by Viro [42], Mazurovskiĭ [34], and Borobia and Mazurovskiĭ [2]. Clearly, if $\mathcal{A}$ is rigidly isotopic to $\mathcal{A}'$, or to its mirror image, then $G(\mathcal{A}) \cong G(\mathcal{A}')$. The converse was established in [32], for $n \leq 6$, using certain invariants derived from the characteristic varieties $V_d(G, \mathbb{C})$ to distinguish the homotopy types of the complements.

We now recover the homotopy-type classification from [32], extending it from 2-arrangements of at most 6 planes to horizontal arrangements of 7 planes, by means of a pair of suitably chosen metabelian Hall invariants.

Theorem 10.6. *For the class of 2-arrangements of $n \leq 7$ planes in $\mathbb{R}^4$ (horizontal if $n = 7$), the rigid isotopy-type classification, up to mirror images, coincides with the isomorphism-type classification of the fundamental groups. For $n \leq 6$, the groups are classified by the Hall invariant δ_{A_4}; for $n = 7$, the Hall invariant δ_{S_3} is also needed.*

Proof. The proof is based on the computations displayed in Table 2. The first column lists the rigid isotopy classes (mirror pairs identified) of 2-arrangements $\mathcal{A}$ (with $n = |\mathcal{A}| \leq 6$, or $n = 7$ and $\mathcal{A}$ horizontal), according to the classification by Viro, Mazurovskiĭ, and Borobia [42, 34, 2]. The next two columns list the Hall invariants δ_{S_3} and δ_{A_4} for the corresponding 2-arrangement groups, $G = G(\mathcal{A})$. These invariants are computed from the varieties $V_d(G, \mathbb{F}_3)$ and $V_d(G, \mathbb{F}_4)$, using Theorem 7.7, as in Examples 10.2 and 10.3. □

$\mathcal{A}$	δ_{S_3}	δ_{A_4}	$\delta_{M_{3,7}}$	$\mathcal{A}$	δ_{S_3}	δ_{A_4}	$\delta_{M_{3,7}}$
$\mathcal{A}(123)$	3	4	8	$\mathcal{A}(2143567)$	5,436	31,541	125,760
$\mathcal{A}(1234)$	28	65	208	$\mathcal{A}(2154367)$	5,112	25,709	83,928
$\mathcal{A}(2134)$	25	38	72	$\mathcal{A}(2165437)$	5,274	31,541	194,976
$\mathcal{A}(12345)$	195	840	4,560	$\mathcal{A}(3216547)$	4,950	25,709	145,080
$\mathcal{A}(21345)$	168	435	1,184	$\mathcal{A}(2143657)$	4,680	16,961	49,872
$\mathcal{A}(21435)$	150	273	632	$\mathcal{A}(3412567)$	4,464	20,606	74,640
$\mathcal{A}(31425)$	139	191	290	$\mathcal{A}(3125467)$	4,572	16,961	59,952
$\mathcal{A}(123456)$	1,240	10,285	96,800	$\mathcal{A}(4123657)$	4,464	16,961	72,720
$\mathcal{A}(213456)$	1,051	5,182	23,560	$\mathcal{A}(3126457)$	4,032	11,129	39,432
$\mathcal{A}(321456)$	997	4,210	12,640	$\mathcal{A}(3254167)$	4,032	12,587	41,160
$\mathcal{A}(215436)$	889	2,752	9,940	$\mathcal{A}(3142567)$	4,237	15,227	66,330
$\mathcal{A}(214356)$	907	2,752	7,288	$\mathcal{A}(3142657)$	3,796	9,197	25,974
$\mathcal{A}(312546)$	799	1,780	5,008	$\mathcal{A}(3145267)$	3,931	11,651	43,298
$\mathcal{A}(341256)$	799	2,023	5,200	$\mathcal{A}(3415267)$	3,796	10,175	34,410
$\mathcal{A}(314256)$	750	1,474	3,688	$\mathcal{A}(3154267)$	3,850	10,751	34,410
$\mathcal{A}(241536)$	704	1,152	2,368	$\mathcal{A}(2415367)$	3,727	9,349	26,604
$\mathcal{L}$	769	1,631	4,288	$\mathcal{A}(2415637)$	3,619	8,709	26,532
$\mathcal{M}$	685	1,126	1,900	$\mathcal{A}(2516347)$	3,484	7,459	20,452
$\mathcal{A}(1234567)$	7,623	124,124	2,039,128	$\mathcal{A}(3625147)$	3,245	6,349	15,736
$\mathcal{A}(2134567)$	6,408	62,159	488,568	$\mathcal{A}(4136257)$	3,329	6,189	15,082
$\mathcal{A}(3214567)$	5,922	47,579	210,024	$\mathcal{A}(5264137)$	3,417	6,819	15,184

TABLE 2. Hall invariants of groups of 2-arrangements in $\mathbb{R}^4$.

Note that the classification can also be achieved by the Hall invariant $\delta_{M_{3,7}}$ (given in the last column of Table 2), either singly (for $n \leq 6$), or together with δ_{S_3} or δ_{A_4} (for $n = 7$ and $\mathcal{A}$ horizontal). Nevertheless, no combination of these 3 invariants is enough to classify the groups of non-horizontal arrangements of $n = 7$ planes. It would be interesting to know whether other Hall invariants can distinguish those 13 groups.

References

[1] D. Arapura, *Geometry of cohomology support loci for local systems* I, J. Alg. Geom. **6** (1997), 563–597.

[2] A. Borobia, V. Mazurovskiĭ, *Nonsingular configurations of* 7 *lines of* $\mathbb{RP}^3$, J. Knot Theory Ramifications **6** (1997), 751–783.

[3] G. Bredon, *Sheaf theory*, Second edition, Grad. Texts in Math., vol. 170, Springer-Verlag, New York, 1997.

[4] K. S. Brown, *Cohomology of groups*, Corrected reprint of the 1982 original, Grad. Texts in Math., vol. 87. Springer-Verlag, New York, 1994.

[5] ———, *The coset poset and probabilistic zeta function of a finite group*, J. Algebra **225** (2000), 989–1012.

[6] C. Bushnell, I. Reiner, *Zeta functions of arithmetic orders and Solomon's conjectures*, Math. Z. **173** (1980), 135–161.

[7] J. Cogolludo, *Topological invariants of the complements to rational arrangements*, Ph.D. thesis, Univ. of Illinois at Chicago and Univ. Complutense, Madrid, 1999.

[8] D. Cohen, A. Suciu, *The braid monodromy of plane algebraic curves and hyperplane arrangements*, Comment. Math. Helvetici **72** (1997), 285–315.

[9] ———, *Characteristic varieties of arrangements*, Math. Proc. Cambridge Phil. Soc. **127** (1999), 33–53.

[10] C. Curtis, I. Reiner, *Methods of representation theory. Vol. 1* (reprint of the 1981 edition), John Wiley & Sons, New York, 1990.

[11] D. Eisenbud, *Commutative algebra with a view towards algebraic geometry*, Grad. Texts in Math., vol. 150, Springer-Verlag, New York, 1995.

[12] M. Falk, *Arrangements and cohomology*, Ann. Combin. **1** (1997), 135–157.

[13] R. H. Fox, *Free differential calculus. III. Subgroups*, Ann. of Math. **64** (1956), 407–419.

[14] ———, *Metacyclic invariants of knots and links*, Canad. J. Math. **22** (1970), 193–201.

[15] The GAP Group, *GAP–Groups, Algorithms, and Programming, Version 4.1*, Aachen, St. Andrews, 1999; available at `http://www-gap.dcs.st-and.ac.uk/~gap`.

[16] D. Grayson, M. Stillman, *Macaulay 2: a software system for algebraic geometry and commutative algebra*; available at `http://www.math.uiuc.edu/Macaulay2`.

[17] M. Hall, *Subgroups of finite index in free groups*, Canad. J. Math **1** (1949), 187–190.

[18] P. Hall, *The Eulerian functions of a group*, Quart. J. Math **7** (1936), 134–151.

[19] J. Hempel, *Homology of coverings*, Pacific J. Math. **112** (1984), 83–113.

[20] ———, *Homology of branched coverings of 3-manifolds*, Canad. J. Math. **44** (1992), 119–134.

[21] E. Hironaka, *Alexander stratifications of character varieties*, Annales de l'Institut Fourier (Grenoble) **47** (1997), 555–583.

[22] J. H. Kwak, J.-H Chun, J. Lee, *Enumeration of regular graph coverings having finite abelian covering transformation groups*, SIAM J. Discrete Math. **11** (1998), 273–285.

[23] A. Libgober, *On the homology of finite abelian coverings*, Topology. Appl. **43** (1992), 157–166.

[24] ———, *Characteristic varieties of algebraic curves*, `arXiv:math.AG/9801070`.

[25] ———, *First order deformations for rank one local systems with non vanishing cohomology*, Topology Appl. (to appear).

[26] A. Libgober, S. Yuzvinsky, *Cohomology of the Orlik-Solomon algebras and local systems*, Compositio Math. **21** (2000), 337–361.

[27] D. Lind, *A zeta function for $\mathbb{Z}^d$-actions*, In: Ergodic theory of $\mathbb{Z}^d$ actions (Warwick, 1993–1994), London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press, Cambridge, 1996, pp. 433–450.

[28] C. Livingston, *Lifting representations of knot groups*, J. Knot Theory Ramifications **4** (1995), 225–234.

[29] A. Lubotzky, *Counting finite index subgroups*, In: Groups '93, Galway/St. Andrews, Vol. 2, London Math. Soc. Lecture Note Ser., vol. 212, Cambridge Univ. Press, Cambridge, 1995, pp. 368–404.

[30] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition. With contributions by A. Zelevinsky, Oxford Math. Monographs, Oxford Univ. Press, New York, 1995.

[31] D. Matei, *Fundamental groups of links and arrangements: characteristic varieties, resonance varieties, and finite index subgroups*, Ph.D. thesis, Northeastern Univ., Boston, MA, 1999.

[32] D. Matei, A. Suciu, *Homotopy types of complements of* 2*-arrangements in* $\mathbb{R}^4$, Topology **39** (2000), 61–88.

[33] ———, *Cohomology rings and nilpotent quotients of real and complex arrangements*, In: Arrangements–Tokyo 1998 (M. Falk, H. Terao, eds.), Adv. Stud. Pure Math., vol. 27, Math. Soc. Japan, Kinokuniya, Tokyo, 2000, pp. 185–215.

[34] V. Mazurovskiĭ, *Configurations of six skew lines* (Russian), Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI) **167** (1988), 121–134; translation in J. Soviet Math. **52** (1990), 2825–2832.

[35] A. D. Mednykh, *Unramified coverings of compact Riemann surfaces* (Russian), Dokl. Akad. Nauk SSSR **244** (1979), 529–532; English translation: Soviet Math. Dokl. **20** (1979), 85–88.

[36] R. Penne, *The Alexander polynomial of a configuration of skew lines in* 3*-space*, Pacific J. Math. **186** (1998), 315–348.

[37] D. Robinson, *A course in the theory of groups*, Grad. Texts in Math., vol. 80, Springer-Verlag, New York, 1996.

[38] S. Roman, *Field theory*, Grad. Texts in Math., vol. 158, Springer-Verlag, New York, 1995.

[39] M. Sakuma, *Homology of abelian coverings of links and spatial graphs*, Canad. J. Math. **47** (1995), 201–224.

[40] R. Stanley, *Enumerative combinatorics. Vol.* 2, Cambridge Studies in Advanced Math., vol. 62, Cambridge Univ. Press, Cambridge, 1999.

[41] A. Suciu, *Translated tori in the characteristic varieties of complex hyperplane arrangements*, Topology. Appl. (to appear), `arXiv:math.AG/9912227`.

[42] O. Viro, *Topological problems concerning lines and points of three-dimensional space*, Dokl. Akad. Nauk. SSSR **284** (1985), 1049–1052; English transl., Soviet Math. Dokl. **32** (1985), 528–531.

[43] L. Weisner, *Some properties of prime-power groups*, Trans. Amer. Math. Soc. **38** (1935), 485–492.

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